

Maintaining Matroid 3-Connectivity
With Respect to a Fixed Basis

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Abstract

We show that for any 3-connected matroid M on a ground set of at least four elements such that M does not contain any 4-element fans, and any basis B of M , there exists a set $K \subseteq E(M)$ of four distinct elements such that for all $k \in K$, $\text{si}(M/k)$ is 3-connected whenever $k \in B$, and $\text{co}(M \setminus k)$ is 3-connected whenever $k \in E(M) - B$. Moreover, we show that if no other elements of $E(M) - K$ satisfy this property, then M necessarily has path-width 3.

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Chapter 1

Introduction

It is well established that the class of 3-connected matroids is fundamental in matroid theory. Indeed, any matroid that is not 3-connected can be constructed from a collection of its 3-connected minors by a sequence of direct sums and 2-sums. Moreover, when working with representable matroids, 3-connectivity is an essential ingredient in obtaining any sense of uniqueness of representations.

Bixby's Lemma [3] (see Lemma 3.4.2) is a useful tool for maintaining matroid 3-connectivity. It states that for any element e of a 3-connected matroid M , either M/e is 3-connected up to parallel classes, or $M \setminus e$ is 3-connected up to series classes. However, situations arise where we have some particular type of element of a 3-connected matroid, and we specifically want to contract it (or specifically want to delete it) whilst maintaining 3-connectivity. In such situations, Bixby's Lemma may be of little use.

For example, if we have a representation $[I_r, D]$ of a matroid M in standard form, then the columns of I_r correspond to a basis B of M , while the columns of D correspond to a cobasis $E(M) - B$ of M . We can now suppress mention of the identity submatrix I_r , and take D to be our representation of M ; where rows of D correspond to the basis B and columns of D to the cobasis $E(M) - B$. Suppose then that we want to study representations of minors of M . If we restrict ourselves to those minors that can be obtained via contracting elements of the basis B or deleting elements of the cobasis $E(M) - B$, then a representation for such a minor can be found simply by deleting rows or columns of the matrix D . Doing so leaves all remaining

entries of D unchanged, so that important structural information that may be present in D can be preserved.

Oxley, Semple, and Whittle [10] studied this precise situation. One result of theirs was that for a 3-connected matroid M with no 4-element fans, and any basis B , there always exists either some element $b \in B$ such that M/b is 3-connected, or some element $b^* \in E(M) - B$ such that $M \setminus b^*$ is 3-connected. In this study, we were interested in extending this result in order to obtain a set K of as many elements as possible such that whenever $e \in K \cap B$, M/e is 3-connected up to parallel classes, and whenever $e \in K \cap (E(M) - B)$, $M \setminus e$ is 3-connected up to series classes. In Theorem 4.0.1, we show that provided $|E(M)| \geq 4$, M always contains at least four such elements. We then looked at situations where this lower bound is met; and concluded with the result that in such a case, the matroid M must have path-width 3 (Theorem 5.0.2).

While the focus of this work was predominantly on the original research undertaken, I have attempted to make the thesis as self contained as possible. Definitions are given for all terminology and notation used, and proofs are provided for all results that are utilised. We use the notation \subseteq to mean subset and \subset for proper subset. Chapters 2 and 3 give a short introduction to matroid theory and matroid connectivity. Chapter 4 is concerned with finding a set consisting of elements which can be removed whilst maintaining 3-connectivity up to parallel and series classes; and culminates with Theorem 4.0.1. In Chapter 5, we study those situations where exactly four such elements are found. Within this chapter, Sections 5.1 and 5.2 present results on how we may arrange certain key elements and substructures present in our matroids with respect to 3-separations. Section 5.3 then ties these results in with the notion of path-width; concluding with Theorem 5.0.2.

As we neared a completed proof of Theorem 5.0.2, the fog of war that had surrounded the problem lifted greatly. It is now clear that there is a slightly more efficient route that can be used to prove the result. In particular, some constructions present in Section 5.1 can be largely circumvented. Nevertheless, we have opted to include and use them, as they do give some insight into the possible structures that can arise. We defer the use of the simplified argument to a research paper.

With regards to the originality of the material presented herein, Chapter 2 consists exclusively of background material on matroid theory, and as such, is all well known. Likewise for Chapter 3; with the exception of 3.4.4 - 3.4.8;

which are all original, and 3.5.1; where a new definition is made. Everything in Chapter 4 is original work. Theorem 5.0.2 is new, as is the material contained in Sections 5.1 and 5.2. Definition 5.3.1 is given in [5], and the results of Lemmas 5.3.2 -5.3.4 are known. The remainder of Section 5.3 is original work.

Chapter 2

Matroids

The purpose of this chapter is to define matroids and take a look at some of their basic properties. The introduction is brief, and material presented herein has been chosen according to what background knowledge is explicitly required for an understanding of the thesis as a whole. For a thorough introduction to matroid theory, see *Matroid Theory* by Oxley [9].

2.1 Matroids; Independent Sets

Matroids can be defined by means of a number of equivalent axiom schemes. The definition we give here is, arguably, perhaps the most intuitive.

Definition 2.1.1. A *matroid* is a pair (E, \mathcal{I}) where E is a finite set, and $\mathcal{I} \subseteq \mathcal{P}(E)$ satisfying the following axioms:

- (I1) $\emptyset \in \mathcal{I}$.
- (I2) If $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$.
- (I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there exists some $e \in I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

If (E, \mathcal{I}) is a matroid, we say that (E, \mathcal{I}) is a matroid *on* E . E is the *ground set* of the matroid. The members of \mathcal{I} are the *independent sets* of the matroid.

Members of $\mathcal{P}(E) - \mathcal{I}$ are called *dependent sets*. We usually simply denote matroids by a capital letter such as M , and whenever we wish to explicitly refer to either the ground set of M or the collection of independent sets of M , we use the notation $E(M)$ or $\mathcal{I}(M)$ respectively.

Definition 2.1.2. A *basis* of a matroid M is a maximal independent set of M .

Lemma 2.1.3. [9, Lemma 1.2.1] *If B_1 and B_2 are bases of a matroid M , then $|B_1| = |B_2|$.*

Proof. Suppose B_1, B_2 are bases of some matroid M where $|B_1| < |B_2|$. Then by **(I3)**, there exists some $e \in B_2 - B_1$ such that $B_1 \cup \{e\}$ is independent. This contradicts the maximality of B_1 . Hence $|B_1| \geq |B_2|$, and similarly, $|B_2| \geq |B_1|$. \square

Lemma 2.1.4. [9, Lemma 1.2.2] *Let B_1 and B_2 be bases of a matroid M and $x \in B_1 - B_2$. Then there exists some $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\}$ is a basis of M .*

Proof. Both $B_1 - \{x\}$ and B_2 are independent, and by Lemma 2.1.3, $|B_1 - \{x\}| < |B_2|$. Therefore, by **(I3)**, there exists some $y \in B_2 - (B_1 - \{x\})$ such that $(B_1 - \{x\}) \cup \{y\}$ is independent. Now $(B_1 - \{x\}) \cup \{y\}$ is contained in some basis of M , and $|(B_1 - \{x\}) \cup \{y\}| = |B_1|$, so that by Lemma 2.1.3, $(B_1 - \{x\}) \cup \{y\}$ is itself a basis of M . \square

Definition 2.1.5. A *circuit* of a matroid M is a minimal dependent set. Circuits which consist of a single element are called *loops*, two element circuits are referred to as a *parallel pair*, and a circuit of size three is a *triangle*.

Lemma 2.1.6. [9, Lemma 1.1.3] *Let C_1 and C_2 be distinct circuits of a matroid M and $e \in C_1 \cap C_2$. Then there exists a circuit C_3 of M such that $C_3 \subseteq (C_1 \cup C_2) - \{e\}$.*

Proof. Suppose $(C_1 \cup C_2) - \{e\}$ is independent. Taking any $x \in C_2 - C_1 \neq \emptyset$, we have that $C_2 - \{x\}$ is independent. Now let I be a maximal independent set of $C_1 \cup C_2$ such that I contains $C_2 - \{x\}$. Then $x \notin I$ and there exists some $y \in C_1 - I$. Therefore

$$|I| \leq |(C_1 \cup C_2) - \{x, y\}| = |C_1 \cup C_2| - 2 < |(C_1 \cup C_2) - \{e\}|.$$

Now applying **(I3)**, we conclude that there must exist an independent I' such that $C_2 - \{x\} \subseteq I' \subseteq C_1 \cup C_2$ and $|I'| > |I|$. This contradicts the maximality of I . \square

Lemma 2.1.7. *Let B_1 and B_2 be bases of a matroid M and $x \in B_2 - B_1$. Then there exists some $y \in B_1 - B_2$ such that $(B_1 - \{y\}) \cup \{x\}$ is a basis of M .*

Proof. As $B_1 \cup \{x\}$ is dependent, it must contain a circuit C_1 . Take any $y \in C_1 - B_2 \neq \emptyset$ and consider $(B_1 - \{y\}) \cup \{x\}$. If $(B_1 - \{y\}) \cup \{x\}$ contains a circuit C_2 , then by Lemma 2.1.6, there exists a circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2) - \{x\} \subseteq B_1$, but this contradicts the fact that B_1 is independent. Therefore $(B_1 - \{y\}) \cup \{x\}$ must be independent. As $|(B_1 - \{y\}) \cup \{x\}| = |B_1|$, it then follows from Lemma 2.1.3 that $(B_1 - \{y\}) \cup \{x\}$ is a basis of M . \square

2.2 Representable Matroids

Matroids were originally introduced by Whitney in 1935 [16]; largely in an attempt to capture the notion of linear independence present in vector spaces in an axiomatic manner.

Proposition 2.2.1. *Let \mathbb{F} be a field and $n \in \mathbb{Z}^+$. Let E be a multiset of vectors from \mathbb{F}^n . Define*

$$\mathcal{I} = \{I \subseteq E : I \text{ linearly independent}\}.$$

Then (E, \mathcal{I}) is a matroid.

Proof. \mathcal{I} clearly satisfies **(I1)** and **(I2)**. Suppose I_1 and I_2 are linearly independent subsets of E where $|I_1| < |I_2|$. Let $W = \text{Span}(I_1 \cup I_2) \subseteq \mathbb{F}^n$. Letting $\dim(W)$ denote the dimension of W , we have $|I_2| \leq \dim(W)$. Assume that for all $e \in I_2 - I_1$, $I_1 \cup \{e\}$ is linearly dependent. Then $W \subseteq \text{Span}(I_1)$ and

$$|I_2| \leq \dim(W) \leq |I_1| < |I_2|$$

which is absurd. Hence there must exist some $e \in I_2 - I_1$ such that $I_1 \cup \{e\}$ is linearly independent; this gives **(I3)**. \square

Suppose that A is some $n \times p$ matrix over a field \mathbb{F} . Then the collection of columns of A correspond to a multiset of vectors from \mathbb{F}^n . In light of Proposition 2.2.1 above, there is a matroid which has the collection of columns of A as its ground set; in which the independent sets are precisely the sets of columns of A which are linearly independent over \mathbb{F}^n . We write $M[A]$ to represent the matroid obtained from a matrix A in this manner. Such a matroid is said to be *representable*. In particular, if $M = M[A]$ is some matroid obtained from a matrix A over \mathbb{F} , then M is said to be representable over \mathbb{F} .

We see that Whitney [16] did indeed capture the notion of linear independence to some extent. But do matroids correspond identically to those structures which arise from configurations of vectors as in the above proposition? The following example due to Vámos [15] answers this question in the negative; the class of representable matroids is strictly contained within the class of matroids.

Example 2.2.2. Let $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and

$$\mathcal{I} = \{X : X \subset E, |X| \leq 4\} - \\ \{\{1, 2, 3, 4\}, \{1, 4, 5, 6\}, \{1, 4, 7, 8\}, \{2, 3, 5, 6\}, \{2, 3, 7, 8\}\}$$

Then (E, \mathcal{I}) is a matroid: the *Vámos matroid*, which is not representable over any field.

So can we somehow classify those matroids which *are* representable? Perhaps it would be possible to tighten our definition of a matroid as to exclude non-representable structures? As the following example illustrates, we need to specify which particular field \mathbb{F} we are considering when determining whether or not a matroid M is representable.

Example 2.2.3. Let $E = \{a, b, c, d\}$ and $\mathcal{I} = \{X : X \subset E, |X| \leq 2\}$. Then $M = (E, \mathcal{I})$ is readily seen to be a matroid which is representable over \mathbb{R} . However, a simple combinatorial examination reveals that M is *not* representable over $GF(2)$. This particular matroid is denoted $U_{2,4}$.

So fix some field \mathbb{F} . Now given an arbitrary matroid M , how do we decide whether or not M is \mathbb{F} representable? This question is a natural one, and one that has been studied extensively. Yet it remains largely unanswered. In Section 2.7, we shall survey one approach that has been taken in an attempt to solve this problem.

2.3 Rank

Definition 2.3.1. Let $M = (E, \mathcal{I})$ be a matroid. Then for any $X \subseteq E$, the *rank* of X ; denoted $r(X)$, is the size of the largest independent set contained in X . We write $r(M)$ to represent $r(E(M))$: the rank of M . When we are dealing with multiple matroids, we often write $r_M(X)$ to denote the rank of the set X in the matroid M .

It follows from Lemma 2.1.3 that the rank of a matroid is equal to the size of any basis for that matroid.

Lemma 2.3.2. *Let $M = (E, \mathcal{I})$ be a matroid. For any $X, Y \subseteq E$,*

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y).$$

Proof. If $r(X \cup Y) = r(X \cap Y)$, this is immediate. So assume that is not the case. Let A be a maximally sized independent set contained in $X \cap Y$. Let A' be a maximally sized independent set contained in $X \cup Y$. Then $|A| < |A'|$, and so, by repeatedly applying **(I3)**, we can extend A to a maximally sized independent set of $X \cup Y$. Letting B denote this extension of A , we have $r(X \cup Y) = |B|$ and $r(X \cap Y) = |A|$. Now we have

$$\begin{aligned} r(X) + r(Y) &\geq |B \cap X| + |B \cap Y| \\ &= |B \cap (X \cup Y)| + |B \cap X \cap Y| \\ &= |B| + |A| \\ &= r(X \cup Y) + r(X \cap Y). \end{aligned}$$

□

If M is a matroid and $X \subseteq E(M)$ is such that $r(X) = 2$, then we call X a *line* of M . When L is a line of M with $|L| = n$, we often say that L is a *n -point line*. Note that some authors take lines of matroids to be rank 2 flats. In this thesis, lines are simply rank 2 sets, and need not be flats. We present two simple results about lines of matroids:

Lemma 2.3.3. *Let M be a matroid containing a n -point line L , and let B be a basis of M . Then $|L \cap B| \leq 2$.*

Proof. If $|L \cap B| \geq 3$, then $r(L \cap B) \geq 3$ because B is independent. But $r(L \cap B) \leq r(L) = 2$, so $|L \cap B| \leq 2$. \square

Lemma 2.3.4. *Let M be a matroid with 4-point lines $L_1, L_2 \subseteq E(M)$. If $r(L_1 \cap L_2) \geq 2$, then either:*

- (i) $L_1 = L_2$, or
- (ii) $L_1 \cup L_2$ is a line with $|L_1 \cup L_2| \geq 5$.

Proof. Suppose $r(L_1 \cap L_2) \geq 2$ and $L_1 \neq L_2$. Then certainly, $|L_1 \cup L_2| \geq 5$, and, by Lemma 2.3.2, $r(L_1 \cup L_2) \leq r(L_1) + r(L_2) - r(L_1 \cap L_2) = 2$. \square

If $M = M[A]$ is some representable matroid, it follows from elementary linear algebra that the rank of M corresponds identically to the dimension of the row space of the matrix A : the number of non-zero rows in a reduced row echelon form of A .

2.4 Closure

Definition 2.4.1. Let M be a matroid. Then for all $X \subseteq E(M)$, the *closure* of X ; denoted $\text{cl}(X)$ is defined by

$$\text{cl}(X) = \{x \in E(M) : r(X \cup \{x\}) = r(X)\}.$$

A set $X \subseteq E(M)$ is *closed* if $\text{cl}(X) = X$. If $X \subseteq E(M)$ is a closed set with $r(X) = r(M) - 1$, then we refer to X as a *hyperplane* of M . We say that $Y \subseteq E(M)$ is *spanning* if $\text{cl}(Y) = E(M)$. When dealing with multiple matroids, we often write $\text{cl}_M(X)$ to denote the closure of X in the matroid M .

The following result is used freely throughout this thesis.

Lemma 2.4.2. *Let M be a matroid and $X \subseteq E(M)$. Then X is a circuit if and only if X is a minimal set with the property that for all $x \in X$, $x \in \text{cl}(X - \{x\})$.*

Proof. X is a circuit iff X is a minimal dependent set; iff X is dependent and for all $x \in X$, $X - \{x\}$ is independent; iff X is a minimal set such that for all $x \in X$, $r(X - \{x\}) = r(X)$; iff X is a minimal set with the property that for all $x \in X$, $x \in \text{cl}(X - \{x\})$. \square

2.5 Duality

For any matroid M with ground set E , there is a unique dual matroid M^* on E defined as follows.

Proposition 2.5.1. *Let $M = (E, \mathcal{I})$ be a matroid, and let \mathcal{B} denote the collection of bases of M . Let*

$$\mathcal{B}^* = \{E - B : B \in \mathcal{B}\}, \text{ and}$$

$$\mathcal{I}^* = \{I \subseteq E : I \subseteq B^* \text{ for some } B^* \in \mathcal{B}^*\}$$

Then $M^ = (E, \mathcal{I}^*)$ is a matroid.*

Proof. As $\mathcal{I} \neq \emptyset$, $\mathcal{B} \neq \emptyset$, so $\mathcal{B}^* \neq \emptyset$ and $\mathcal{I}^* \neq \emptyset$; hence **(I1)** is satisfied. Suppose $I_1 \in \mathcal{I}^*$. Then $I_1 \subseteq B_1^*$ for some $B_1^* \in \mathcal{B}^*$. Therefore for any $I_2 \subseteq I_1$, $I_2 \subseteq B_1^*$ also, so that $I_2 \in \mathcal{I}^*$; which confirms that **(I2)** holds. To show that $M^* = (E, \mathcal{I}^*)$ satisfies **(I3)**, we first note the following:

2.5.1.1. *If $B_1^*, B_2^* \in \mathcal{B}^*$ and $x \in B_1^* - B_2^*$, then there exists some $y \in B_2^* - B_1^*$ such that $(B_1^* - \{x\}) \cup \{y\} \in \mathcal{B}^*$.*

Proof. Suppose $B_1^*, B_2^* \in \mathcal{B}^*$ and $x \in B_1^* - B_2^*$. Then there exist bases $B_1 = B_1^* - E$ and $B_2 = B_2^* - E$ of M such that $x \in B_2 - B_1$. Now by Lemma 2.1.7, there exists some $y \in B_1 - B_2$ such that $(B_1 - \{y\}) \cup \{x\}$ is a basis of M . Therefore $E - ((B_1 - \{y\}) \cup \{x\}) = (B_1^* - \{x\}) \cup \{y\} \in \mathcal{B}^*$. \square

Assume that there exists $I_1, I_2 \in \mathcal{I}^*$, where $|I_1| < |I_2|$ and for all $e \in I_2 - I_1$, $I_1 \cup \{e\} \notin \mathcal{I}^*$. By definition, there exists $B_1^*, B_2^* \in \mathcal{B}^*$ such that $I_1 \subseteq B_1^*$ and $I_2 \subseteq B_2^*$. Let B_1^* be any such element of \mathcal{B}^* , and choose $B_2^* \in \mathcal{B}^*$ such that $|B_2^* - (I_2 \cup B_1^*)|$ is minimal.

Suppose there exists some $x \in B_2^* - (I_2 \cup B_1^*)$. Then by 2.5.1.1, there exists some $y \in B_1^* - B_2^*$ such that $(B_2^* - \{x\}) \cup \{y\} \in \mathcal{B}^*$. But then $|((B_2^* - \{x\}) \cup \{y\}) - (I_2 \cup B_1^*)| < |B_2^* - (I_2 \cup B_1^*)|$; which contradicts our choice of B_2^* . Therefore $B_2^* - (I_2 \cup B_1^*) = \emptyset$, and it follows that $B_2^* - B_1^* = I_2 - B_1^*$, which implies that $B_2^* - B_1^* = I_2 - I_1$.

Now suppose there exists some $x \in B_1^* - (I_1 \cup B_2^*)$. Then by 2.5.1.1, there exists some $y \in B_2^* - B_1^*$ such that $(B_1^* - \{x\}) \cup \{y\} \in \mathcal{B}^*$. Now $I_1 \cup \{y\} \subseteq$

$(B_1^* - \{x\}) \cup \{y\} \in \mathcal{B}^*$. Hence $I_1 \cup \{y\} \in \mathcal{I}^*$. But $B_2^* - B_1^* = I_2 - I_1$, so $y \in I_2$, but this contradicts our assumption that for all $e \in I_2 - I_1$, $I_1 \cup \{e\} \notin \mathcal{I}^*$. Therefore $B_1^* - (I_1 \cup B_2^*) = \emptyset$, which implies that $B_1^* - B_2^* = I_1 - B_2^*$, and so $B_1^* - B_2^* \subseteq I_1 - I_2$.

By Lemma 2.1.3, $|B_1^*| = |B_2^*|$, so $|B_1^* - B_2^*| = |B_2^* - B_1^*|$. But we've seen that $B_2^* - B_1^* = I_2 - I_1$ and $B_1^* - B_2^* \subseteq I_1 - I_2$, therefore $|I_2 - I_1| \leq |I_1 - I_2|$; which contradicts the fact that $|I_2| > |I_1|$. We are left to conclude that there indeed must exist some $e \in I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}^*$; which proves **(I3)**. Therefore $M^* = (E, \mathcal{I}^*)$ is a matroid. \square

Let M^* be a matroid obtained from a matroid M as in the above proposition. Then M^* is said to be the *dual* of M . A basis of M^* is called a *cobasis* of M . Similarly, independent sets, circuits, hyperplanes, spanning sets and lines of M^* are referred to as *coindependent sets*, *cocircuits*, *cohyperplanes*, *cospansing sets*, and *colines* of M respectively. Whenever Γ is a coline of a matroid with $|\Gamma| = n$, we say that Γ is a *n-point coline*. One, two, and three element cocircuits of M are called *coloops*, *series pairs* and *triads* respectively. Note also that $(M^*)^* = M$.

Lemma 2.5.2. *Let M be a matroid on a ground set E , and $X \subseteq E$. Then*

- (i) *X is coindependent if and only if $E - X$ is spanning, and*
- (ii) *X is a cohyperplane if and only if $E - X$ is a circuit.*

Proof. X is coindependent iff X is contained in a cobasis; iff $E - X$ contains a basis; iff $r(E - X) = r(M)$; iff $\text{cl}(E - X) = E$; iff $E - X$ is spanning. It then follows that X is a cohyperplane iff X is a maximal non-cospansing set; iff $E - X$ is a minimal dependent set; iff $E - X$ is a circuit. \square

Let M be a matroid. For any $X \subseteq E(M)$, the *corank* of X ; denoted $r^*(X)$ is equal to $r_{M^*}(X)$. The *coclosure* of X ; denoted $\text{cl}^*(X)$ is equal to $\text{cl}_{M^*}(X)$. A set X is *coclosed* if $\text{cl}^*(X) = X$.

The corank function of a matroid satisfies the following:

Lemma 2.5.3. *Let M be a matroid on a ground set E . For any $X \subseteq E$, $r^*(X) = |X| - r(M) + r(E - X)$.*

Proof. Take any $X \subseteq E$. Let I^* be a maximally sized coindependent set contained in X . By Lemma 2.5.2, $E - I^*$ is spanning. Hence $E - I^*$ must contain a basis B . It then follows that $X \cap B = X - I^*$.

If $r(E - X) \geq |B - X|$, then by **(I3)**, there exists an element $e \in E - (X \cup B)$ such that $(B - X) \cup \{e\}$ is independent. Therefore $(B - X) \cup \{e\}$ is contained in some basis B' . But then $X - B'$ is coindependent by Lemma 2.5.2 and $|X - B'| > |X - B| = |I^*|$; which contradicts our choice of I^* . So we must have $r(E - X) = |B - X|$, and so:

$$\begin{aligned} |X| &= |X \cap I^*| + |X \cap B| \\ &= r^*(X) + |X \cap B| \\ &= r^*(X) + |B| - |B - X| \\ &= r^*(X) + r(M) - |B - X| \\ &= r^*(X) + r(M) - r(E - X). \end{aligned}$$

□

Lemma 2.5.4. *Let e be an element of a matroid M , and let X and Y be disjoint sets whose union is $E(M) - \{e\}$. Then $e \in \text{cl}(X)$ if and only if $e \notin \text{cl}^*(Y)$.*

Proof. By Lemma 2.5.3, $r^*(Y) = |Y| - r(M) + r(X \cup \{e\})$ and $r^*(Y \cup \{e\}) = |Y| + 1 - r(M) + r(X)$. Therefore $r^*(Y \cup \{e\}) = r^*(Y) + 1 + (r(X) - r(X \cup \{e\}))$, so that $e \in \text{cl}(X)$ iff $r^*(Y) \neq r^*(Y \cup \{e\})$; iff $e \notin \text{cl}^*(Y)$. □

Lemma 2.5.5. *Let M be a matroid containing a circuit C and a cocircuit C^* . Then $|C \cap C^*| \neq 1$.*

Proof. Suppose that $C \cap C^* = \{x\}$. Letting $H = E(M) - C^*$, it follows from Lemma 2.5.2 that H is a hyperplane. Now $C - \{x\} \subseteq H$, and so, by Lemma 2.4.2, $x \in \text{cl}(C - \{x\}) \subseteq \text{cl}(H)$. But then $x \in \text{cl}(H) - H$; which contradicts the fact that hyperplanes are closed. Therefore $|C \cap C^*| \neq 1$. □

Duality is a powerful tool in matroid theory. Whenever we prove that some result must hold for a matroid M , we may dualise the statement of that result to immediately obtain another result that holds for the matroid M^* . The following two facts about colines; which will be useful in what follows, are a good illustration of this.

Lemma 2.5.6. *Let M be a matroid containing an n -point coline Γ , and let B be a basis of M . Then $|\Gamma \cap B| \geq n - 2$.*

Proof. By dualising Lemma 2.3.3, we have the following result: If M is a matroid containing an n -point coline L , and B is a cobasis of M , then $|L \cap B| \leq 2$. Equivalently: if M is a matroid containing an n -point coline Γ , and $E(M) - B$ is a basis of M , then $|L \cap (E(M) - B)| \geq n - 2$. The result then follows via a relabelling of the basis $E(M) - B$. \square

Lemma 2.5.7. *Let M be a matroid with 4-point colines $\Gamma_1, \Gamma_2 \subseteq E(M)$. If $r^*(\Gamma_1 \cap \Gamma_2) \geq 2$, then either:*

- (i) $\Gamma_1 = \Gamma_2$, or
- (ii) $\Gamma_1 \cup \Gamma_2$ is a coline with $|\Gamma_1 \cup \Gamma_2| \geq 5$.

Proof. This follows immediately by dualising Lemma 2.3.4, and relabelling so that L_1 and L_2 become Γ_1 and Γ_2 respectively. \square

We now note that if A is a representation of a matroid over some field \mathbb{F} , then the underlying matroid is invariant under applying standard row operations to the matrix A , as well as being invariant upon interchanging columns within A , and multiplying a column of A by an element of \mathbb{F} . From this it is readily deduced that for a matroid M representable over \mathbb{F} , the dual matroid M^* is also representable over \mathbb{F} , and indeed, we can explicitly construct a representation of M^* . We say that the class of \mathbb{F} -representable matroids is closed under duality.

2.6 Minors

We now briefly examine the fundamental substructures of a matroid: its minors. One natural substructure of a matroid M is found when we simply

shrink the ground set $E(M)$ and observe the restriction of M on this smaller ground set. More formally:

Definition 2.6.1. Let $M = (E, \mathcal{I})$ be a matroid and $e \in E$. Let

$$\mathcal{I}(M \setminus e) = \{I \subseteq E - \{e\} : I \in \mathcal{I}\}.$$

Then $(E, \mathcal{I}(M \setminus e))$ is readily seen to be a matroid; called M delete e , and denoted $M \setminus e$. We say that $M \setminus e$ is obtained from M by *deleting* e .

Armed with duality, we explore another natural matroid substructure.

Definition 2.6.2. Let M be a matroid and $e \in E(M)$. The matroid M contract e ; denoted M/e is defined to be $M/e = (M^* \setminus e)^*$. M/e is said to be obtained from M by *contracting* e .

Lemma 2.6.3. Suppose that M is a matroid on a ground set E , $e \in E$ and $X \subseteq E - \{e\}$. Then $r_{M/e}(X) = r_M(X \cup \{e\}) - r_M(\{e\})$.

Proof. $r_{M/e}(X) = r_{(M^* \setminus e)^*}(X)$, so, by Lemma 2.5.3,

$$\begin{aligned} r_{M/e}(X) &= |X| - r(M^* \setminus e) + r_{M^* \setminus e}(E - (X \cup \{e\})) \\ &= |X| - r^*(E - \{e\}) + r^*(E - (X \cup \{e\})) \\ &= |X| - |E - \{e\}| - r_M(\{e\}) + |E - (X \cup \{e\})| + r_M(X \cup \{e\}) \\ &= |X| + 1 - |X \cup \{e\}| + r_M(X \cup \{e\}) - r_M(\{e\}) \\ &= r_M(X \cup \{e\}) - r_M(\{e\}). \end{aligned}$$

□

Lemma 2.6.4. Let M be a matroid with distinct $e_1, e_2 \in E(M)$. Then

- (i) $(M \setminus e_1) \setminus e_2 = (M \setminus e_2) \setminus e_1$, and
- (ii) $(M/e_1)/e_2 = (M/e_2)/e_1$, and
- (iii) $(M \setminus e_1)/e_2 = (M/e_2) \setminus e_1$.

Proof. A set $I \subseteq E(M)$ is independent in $(M \setminus e_1) \setminus e_2$ iff $I \subseteq E(M) - \{e_1, e_2\}$ and I is independent in M ; iff I is independent in $(M \setminus e_2) \setminus e_1$; which proves (i). Now by the definition of contraction together with (i), we have

$$\begin{aligned} (M/e_1)/e_2 &= (M^* \setminus e_1)^*/e_2 = ((M^* \setminus e_1) \setminus e_2)^* = ((M^* \setminus e_2) \setminus e_1)^* \\ &= ((M^* \setminus e_2)^*/e_1 = ((M^*)^*/e_2)/e_1 = (M/e_2)/e_1 \end{aligned}$$

so that (ii) holds. To prove (iii), take any $X \subseteq E(M) - \{e_1, e_2\}$. Then by Lemma 2.6.3,

$$\begin{aligned} r_{(M/e_2) \setminus e_1}(X) &= r_{M/e_2}(X) \\ &= r_M(X \cup \{e_2\}) - r_M(\{e_2\}) \\ &= r_{M \setminus e_1}(X \cup \{e_2\}) - r_{M \setminus e_1}(\{e_2\}) \\ &= r_{(M \setminus e_1)/e_2}(X). \end{aligned}$$

Therefore the rank in $(M \setminus e_1)/e_2$ of any subset $X \subseteq E(M) - \{e_1, e_2\}$ is equal to the rank of the same subset in the matroid $(M/e_2) \setminus e_1$. As $E(M) - \{e_1, e_2\}$ is the ground set of both of these matroids, it follows that the collection of independent sets of $(M \setminus e_1)/e_2$ is identical to the collection of independent sets of $(M/e_2) \setminus e_1$; we conclude that $(M \setminus e_1)/e_2 = (M/e_2) \setminus e_1$. \square

As a consequence of the above result we see that the operations of deletion and contraction commute with both themselves and one another. Therefore any sequence of deletions and contractions from a matroid M can be written in the form $M \setminus X/Y$ where X and Y are disjoint subsets of the ground set. Matroids obtained from a given matroid M through any such sequence of deletions and contractions are called *minors* of M . If M is a matroid with a minor N where $N \neq M$, then we say that N is a *proper* minor of M . As we shall see in Section 2.7, knowledge of the minors contained in some matroid M can give us important information about some of the necessary properties of M .

The following is an extension of Lemma 2.6.3, which will be useful later on.

Lemma 2.6.5. *Let M be a matroid and $T \subseteq E(M)$. Then for all $X \subseteq E(M) - T$, $r_{M/T}(X) = r_M(X \cup T) - r_M(T)$.*

Proof. Suppose $X \subseteq E(M) - T$ and let $T = \{e_1, \dots, e_n\}$. By Lemma 2.6.3, $r_{M/e_1}(X) = r_M(X \cup \{e_1\}) - r_M(\{e_1\})$, and it follows that

$$\begin{aligned} r_{M/\{e_1, e_2\}}(X) &= r_{M/e_1}(X \cup \{e_2\}) - r_{M/e_1}(\{e_2\}) \\ &= r_M(X \cup \{e_1, e_2\}) - r_M(\{e_1\}) - r_M(\{e_1, e_2\}) + r_M(\{e_1\}) \\ &= r_M(X \cup \{e_1, e_2\}) - r_M(\{e_1, e_2\}). \end{aligned}$$

Now by an iteration of this argument for $e_i \in \{e_3, \dots, e_n\}$, we can conclude that $r_{M/\{e_1, \dots, e_n\}}(X) = r_M(X \cup \{e_1, \dots, e_n\}) - r_M(\{e_1, \dots, e_n\})$ as required. \square

We conclude this section with a brief discussion on minors of representable matroids. If M is a matroid which is representable over \mathbb{F} , and $e \in E(M)$, then we may obtain a representation for $M \setminus e$ simply by taking a representation of M and deleting the column labelled by e . Also, we noted at the end of Section 2.5 that the class of \mathbb{F} -representable matroids is closed under duality. This implies that for $e \in E(M)$, M/e is also representable over \mathbb{F} . Let A be a representation of M over \mathbb{F} . If e is a loop of M , then $M/e = M \setminus e$, so a representation of M/e is found as above. So assuming that e is not a loop of M , it turns out that to obtain a representation for M/e , we pivot on some non-zero entry of the column of A labelled by e to transform this column into a unit vector. Then, by deleting the row containing the unique non-zero entry of e as well as deleting the column e , we obtain a suitable representation of our minor. It now follows that if N is any minor of M , then N is also \mathbb{F} -representable; the class of \mathbb{F} -representable matroids is *minor-closed*. Moreover, a representation for N can be found algebraically by starting with a representation of M and performing a finite sequence of pivots, row deletions and columns deletions. We shall revisit this in Section 3.5.

2.7 Rota's Conjecture

The class of matroids is partially ordered under the property of being a minor; as can easily be deduced from the definitions in Section 2.6. Also, as discussed above, the class of \mathbb{F} -representable matroids is minor-closed. Now if M is a matroid with $E(M) < 4$, it is a simple exercise to show that M is representable over every field. However, as we saw in Section 2.2, there exist matroids which are representable over some fields but not others, as well as

matroids which are not representable whatsoever. We look to explore the threshold between representability and non-representability. In particular, as the class of \mathbb{F} -representable matroids is minor-closed, we can study those matroids which are minor minimal non-representable matroids over \mathbb{F} . Such minor minimal non-representable matroids are called *excluded minors* for \mathbb{F} . The following result; due to Tutte [14], gives a complete classification of those matroids which are representable over $GF(2)$ in terms of excluded minors. The matroid $U_{2,4}$ was introduced in Example 2.2.3.

Theorem 2.7.1. *A matroid is representable over $GF(2)$ if and only if it has no minor isomorphic to $U_{2,4}$.*

The above is an example of an excluded minor theorem. Such a theorem gives a complete classification of a minor-closed class of matroids with respect to the collection of excluded minors for that class. The following result; due to Lazarsen [7], reveals that for infinite fields, excluded minor theorems need not be as elegant as Theorem 2.7.1.

Theorem 2.7.2. *If \mathbb{F} is a field of characteristic zero, then the set of excluded minors for \mathbb{F} -representability is infinite.*

However, this is in stark contrast to what is generally believed to be true for finite fields. The following conjecture of Rota [12] is perhaps the most well known conjecture in matroid theory.

Conjecture 2.7.3. (Rota's Conjecture) *For any finite field \mathbb{F} , the complete set of excluded minors for \mathbb{F} -representability is finite.*

Proofs that Rota's conjecture is valid in the three element field were first published independently by Bixby [2] and Seymour [13]. The set of excluded minors for $GF(3)$ was found to be as follows:

Theorem 2.7.4. *A matroid is representable over $GF(3)$ if and only if it has no minor isomorphic to any of the matroids $U_{2,5}, U_{3,5}, F_7$, or F_7^* .*

Validity of the conjecture over $GF(4)$ was confirmed by Geelen, Gerards, and Kapoor [4]:

Theorem 2.7.5. *A matroid is representable over $GF(4)$ if and only if it has no minor isomorphic to any of the matroids $U_{2,6}, U_{4,6}, P_6, F_7^-, (F_7^-)^*, P_8$, or P_8'' .*

However, for all finite fields with size exceeding four, Rota's conjecture remains unresolved. Naturally, in seeking a resolution of the conjecture, one would hope to narrow the spectrum of candidates for excluded minor status over $GF(q)$. One simple observation we can make from the above results is that given an arbitrary matroid M , key structural information on M can be obtained by considering the minors of M . Taking some finite field $GF(q)$, this methodology can now be applied directly to the excluded minors for $GF(q)$. We should be able to shrink our list of potential excluded minors for $GF(q)$ by looking at the necessary structural conditions imposed by the minors of such potential excluded minors. In particular, supposing that M is a potential excluded minor for $GF(q)$, any proper minor N of M is representable over $GF(q)$. Supposing that $N_1, \dots, N_k, N_{k+1} = M$ is some ordered sequence of minors of M ; where N_i is a proper minor of N_{i+1} for $i \in \{1, \dots, k\}$, we know that there exists some matrix representation over $GF(q)$ for N_1, \dots, N_k . We can therefore examine the representations themselves and hope to use techniques from linear algebra to place necessary conditions on M through inductive methods. In order to do this, we require a sense of consistency when talking about matroid representations. Loosely speaking, one would ideally want the statement 'the representable matroid N' is a minor of the representable matroid N ' to correspond with the statement 'a representation of the matroid N' can be obtained from a representation of the matroid N through standard algebraic operations' Unfortunately, this is not quite the case.

We noted above that if a matroid M is representable over \mathbb{F} , with some representation A , then M is invariant under applying standard row operations to A , as well as the scaling and interchanging of columns within A . The fact that one may also add and/or remove rows of zeros to/from A , as well as apply some automorphism of the field \mathbb{F} ; neither of which alters $M[A]$, leads us to the following definition.

Let M be some matroid which is representable over \mathbb{F} , and let A_1 and A_2 be matrices; each giving a representation of M over \mathbb{F} . If A_2 can be obtained from A_1 through a sequence of standard row operations, column interchanges, column scalings, addition/removal of rows of zeros, and automorphisms of \mathbb{F} , then the representations A_1 and A_2 are said to be *equivalent*. If all representations of M are equivalent, then we say that M is *uniquely representable*.

The verifications of Rota's conjecture for $GF(2)$, $GF(3)$ and $GF(4)$ all

rely heavily on the fact that for these small fields, the key classes of matroids that need to be considered are classes of uniquely representable matroids. However, Oxley, Vertigan, and Whittle [11] showed that certain classes of matroids representable over $GF(5)$ may have as many as six inequivalent representations, while for finite fields containing greater than five elements, the number of possible inequivalent representations of a matroid can be unbounded. This presents a major obstacle on the road toward a resolution of Rota's conjecture.

This obstacle led Whittle [17] to develop the notion of a stabilizer for a class of matroids. Without indulging deeply into the details of defining a general stabilizer here, essentially, a stabilizer for the class of $GF(q)$ representable matroids is a $GF(q)$ -representable matroid N such that if M is a 3-connected $GF(q)$ -representable matroid which contains N as a minor, then any representation of M is uniquely determined by a representation of N . Therefore, if we are to use representations as an inductive tool for examining properties of matroids, we need to find stabilizers for classes of matroids, study matroids which contain these stabilizers, and, in particular, we need to restrict our attention to matroids which are 3-connected. In what follows, we examine this notion of 3-connectivity.

Chapter 3

Connectivity

3.1 Introduction

A minor N of a matroid M is, by definition, obtained through applying a sequence of deletions and contractions to the matroid M . So if we know that M is 3-connected, and we want to study minors of M that are also 3-connected, we naturally want to know when the fundamental minors $M \setminus e$ and M/e are also 3-connected for particular $e \in E(M)$.

For example, it is a straightforward result that any excluded minor M for $GF(q)$ must be 3-connected. Now, as discussed above, if we want to deduce information about M by examining representations of its minors, then it would certainly help if these minors were uniquely representable; and for that, we require them to be 3-connected. Now if N is a minor of M , then there is a sequence of minors $N = N_1, N_2, \dots, N_k, N_{k+1} = M$ of M where for $i \in \{1, \dots, k\}$, there exists some $e_i \in E(M)$ such that either $N_i = N_{i+1} \setminus e_i$ or $N_i = N_{i+1}/e_i$. Therefore if we can deduce exactly when deleting or contracting an element from a 3-connected matroid results in another 3-connected matroid, we may be able to obtain a sequence of uniquely representable matroids $N = N_1, N_2, \dots, N_k, N_{k+1} = M$; which would be helpful in an attempt to deduce structure of M . Firstly, some definitions:

Definition 3.1.1. Let M be a matroid on a ground set E . The connectivity function λ of M is defined on all subsets X of E by

$$\lambda(X) = r(X) + r(E - X) - r(M).$$

When dealing with multiple matroids, we shall write λ_M to mean the connectivity function of the matroid M . It is clear that $\lambda(X) = \lambda(E - X)$ for any $X \subseteq E$. A subset X or a partition $(X, E - X)$ of M is *k-separating* if $\lambda(X) \leq k - 1$. A *k-separating* partition $(X, E - X)$ is a *k-separation* if both $|X| \geq k$ and $|E - X| \geq k$. A *k-separating* set X , or a *k-separating* partition $(X, E - X)$, or a *k-separation* $(X, E - X)$ is *exact* if $\lambda(X) = k - 1$. A matroid is said to be *n-connected* provided it has no *k-separations* for all $k < n$. When a matroid is 2-connected, we simply say that it is *connected*.

Lemma 3.1.2. *Let M be a matroid. Then $\lambda_M = \lambda_{M^*}$. That is, for all $X \subseteq E(M)$, $\lambda_M(X) = \lambda_{M^*}(X)$.*

Proof. Suppose $X \subseteq E(M)$. Then, by Lemma 2.5.3,

$$\begin{aligned} \lambda_M(X) &= r(X) + r(E - X) - r(M) \\ &= r(X) + r^*(X) - |X| \\ &= r^*(E - X) + r(M) - |E - X| + r^*(X) - |X| \\ &= r^*(X) + r^*(E - X) - (|E(M)| - r(M)) \\ &= r^*(X) + r^*(E - X) - r^*(M) = \lambda_{M^*}(X). \end{aligned}$$

□

The above result, as well as the next two, will be used freely.

Corollary 3.1.3. *Let M be a matroid. Then M is *n-connected* if and only if M^* is *n-connected*.*

Proof. This follows immediately from Lemma 3.1.2. □

Lemma 3.1.4. *Let M be an *n-connected* matroid with $|E(M)| \geq 4$ and $e \in E(M)$. Then both $M \setminus e$ and M/e are $(n - 1)$ -connected.*

Proof. Suppose $N \in \{M \setminus e, M/e\}$ and that X and Y are disjoint subsets of $E(M)$ whose union is $E(M) - \{e\}$. Then by Lemma 2.6.3, $r_N(X) \in \{r_M(X), r_M(X) - 1\}$, $r_N(Y) \in \{r_M(Y), r_M(Y) - 1\}$, and $r(N) \in \{r(M), r(M) - 1\}$. So certainly, $\lambda_N(X) = r_N(X) + r_N(Y) - r(N) \in \{\lambda_M(X), \lambda_M(X) - 1, \lambda_M(X) - 2\}$, and the only permutation that gives $\lambda_N(X) = \lambda_M(X) - 2$ is where $r(N) = r(M)$, $r_N(X) = r_M(X) - 1$, and

$r_N(Y) = r_M(Y) - 1$. But in this case, the fact that $r_N(X) = r_M(X) - 1$ implies that $N = M/e$ and e is not a loop of M ; contradicting the necessary requirement that $r(N) = r(M)$. Therefore $\lambda_N(X) \in \{\lambda_M(X), \lambda_M(X) - 1\}$. As (X, Y) was an arbitrary partition of $E(N)$, it follows that N has no $(n - k)$ -separations for all $k \geq 2$. Therefore N is $(n - 1)$ -connected. \square

Lemma 3.1.5. *Let M be a matroid on a ground set E and suppose $X, Y \subseteq E$. Then $\lambda(X \cup Y) + \lambda(X \cap Y) \leq \lambda(X) + \lambda(Y)$.*

Proof. With the help of Lemmas 2.3.2 and 2.5.3, we have

$$\begin{aligned}
& \lambda(X \cup Y) + \lambda(X \cap Y) \\
&= r(X \cup Y) + r(X \cap Y) + r(E - (X \cup Y)) + r(E - (X \cap Y)) - 2r(M) \\
&\leq r(X) + r(Y) + r(E - (X \cup Y)) + r(E - (X \cap Y)) - 2r(M) \\
&= r(X) + r(Y) + r^*(X \cup Y) + r^*(X \cap Y) - |X \cup Y| - |X \cap Y| \\
&= r(X) + r(Y) + r^*(X \cup Y) + r^*(X \cap Y) - |X| - |Y| \\
&\leq r(X) + r(Y) + r^*(X) + r^*(Y) - |X| - |Y| \\
&= r(X) + r(Y) + r(E - X) + r(E - Y) - 2r(M) = \lambda(X) + \lambda(Y).
\end{aligned}$$

\square

We shall often say *by submodularity* of λ to mean an application of Lemma 3.1.5.

3.2 Various Lemmas on 3-connectivity

We now present a series of results relating specifically to 3-connectivity. The first of these is used repeatedly throughout the thesis. When applying it, we shall often simply use the phrase *by uncrossing*.

Lemma 3.2.1. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$. Then:*

- (i) *if $|X \cap Y| \geq 2$, $X \cup Y$ is 3-separating, and*
- (ii) *if $|E(M) - (X \cup Y)| \geq 2$, $X \cap Y$ is 3-separating.*

Proof. Suppose $|X \cap Y| \geq 2$. Then by submodularity of λ , we have $\lambda(X \cup Y) \leq \lambda(X) + \lambda(Y) - \lambda(X \cap Y) \leq 4 - \lambda(X \cap Y) \leq 2$, so that $X \cup Y$ is 3-separating. Similarly, supposing that $|E(M) - (X \cup Y)| \geq 2$, it follows that $\lambda(X \cap Y) \leq \lambda(X) + \lambda(Y) - \lambda(X \cup Y) \leq 4 - \lambda(E - (X \cup Y)) \leq 2$. \square

Lemma 3.2.2. *Let M be a 3-connected matroid. If L is a line of M with $|L| \geq 4$, then $M \setminus x$ is 3-connected for all $x \in L$.*

Proof. Suppose $x \in L$ and $M \setminus x$ is not 3-connected. Then there exists a 2-separation (P, Q) of $M \setminus x$. Without loss of generality, $|L \cap P| \geq 2$. But now $x \in \text{cl}(P)$, so that $\lambda_{M \setminus x}(P) = \lambda_M(P \cup \{x\})$; giving a contradictory 2-separation $(P \cup \{x\}, Q)$ of the 3-connected matroid M . \square

Lemma 3.2.3. *Let M be a 3-connected matroid. If Γ is a coline of M with $|\Gamma| \geq 4$, then M/y is 3-connected for all $y \in \Gamma$.*

Proof. Let Γ be a coline of M with $|\Gamma| \geq 4$. Then Γ is a line of the 3-connected matroid M^* . So by Lemma 3.2.2, $M^* \setminus y$ is 3-connected for all $y \in \Gamma$. Therefore $(M^* \setminus y)^* = M/y$ is 3-connected for all $y \in \Gamma$. \square

Lemma 3.2.4. *Let M be a 3-connected matroid. Let $\Gamma \subseteq E(M)$ be a coline of M . Then provided $|E(M) - \Gamma| \geq 2$, $r(\Gamma) = |\Gamma|$.*

Proof. Let Γ be a coline of M . Using Lemma 2.5.3 and the fact that M is 3-connected, we have

$$\begin{aligned} \lambda(\Gamma) &= r(\Gamma) + r(E(M) - \Gamma) - r(M) \geq 2 \\ &\implies r(\Gamma) + 2 + r(M) - |\Gamma| - r(M) \geq 2 \\ &\implies r(\Gamma) \geq |\Gamma| \end{aligned}$$

So that $r(\Gamma) = |\Gamma|$. \square

Lemma 3.2.5. *Let M be a 3-connected matroid, and let B be a basis of M . If Γ_1, Γ_2 are 4-point colines of M with $|\Gamma_1 \cap B| = |\Gamma_2 \cap B| = 2$, then, if $\Gamma_1 \cap \Gamma_2 \neq \emptyset$, $\Gamma_1 \cap \Gamma_2 \not\subseteq B$.*

Proof. Let Γ_1, Γ_2 be 4-point colines of M such that $|\Gamma_1 \cap B| = |\Gamma_2 \cap B| = 2$ and $\Gamma_1 \cap \Gamma_2 \neq \emptyset$. As M is 3-connected, it follows from Lemma 2.3.2 that $r^*(\Gamma_1 \cup \Gamma_2) \in \{2, 3\}$. If $\Gamma_1 = \Gamma_2$, then the result is immediate, so assume that

Γ_1 and Γ_2 are distinct. Suppose that $\Gamma_1 \cap \Gamma_2 \subseteq B$. If $|\Gamma_1 \cap \Gamma_2| \geq 2$, then by Lemma 2.5.7, $\Gamma_1 \cup \Gamma_2$ is a coline containing at least five elements. But then $|(\Gamma_1 \cup \Gamma_2) \cap (E(M) - B)| \geq 3$, so that $r^*(\Gamma_1 \cup \Gamma_2) \geq 3$; contradicting the fact that $\Gamma_1 \cup \Gamma_2$ is a coline. Otherwise, $|\Gamma_1 \cap \Gamma_2| = 1$; implying that $|(\Gamma_1 \cup \Gamma_2) \cap (E(M) - B)| \geq 4$; which contradicts the fact $r^*(\Gamma_1 \cup \Gamma_2) \in \{2, 3\}$. Therefore $\Gamma_1 \cap \Gamma_2 \not\subseteq B$. \square

Lemma 3.2.6. *Let (X, Y) be an exact 3-separation of a 3-connected matroid M . If X is independent, then X is a coline of M .*

Proof. Suppose X is independent. Then, by Lemma 2.5.3,

$$\begin{aligned} r^*(X) &= |X| - r(M) + r(E(M) - X) \\ &= |X| - r(M) + r(Y) \\ &= |X| - r(M) + r(M) + 2 - r(X) \\ &= |X| + 2 - r(X) = 2. \end{aligned}$$

So that X is a coline of M . \square

We shall write $x \in \text{cl}^{(*)}(Y)$ to mean that $x \in \text{cl}(Y)$ or $x \in \text{cl}^*(Y)$.

Lemma 3.2.7. *Let X be an exactly 3-separating set in a 3-connected matroid M , and suppose that $r \in E(M) - X$. Then $X \cup \{r\}$ is 3-separating if and only if $r \in \text{cl}^{(*)}(X)$.*

Proof. We have $\lambda(X) = r(X) + r(E(M) - X) - r(M) = 2$. Now Lemma 2.5.4 together with some simple arithmetic reveals that $\lambda(X \cup \{r\}) \leq 2$ iff $r(X \cup \{r\}) + r(E(M) - (X \cup \{r\})) - r(M) \leq 2$; iff $r(X \cup \{r\}) = r(X)$, or $r(X \cup \{r\}) = r(X) + 1$ and $r(E(M) - (X \cup \{r\})) = r(E(M) - X) - 1$; iff $r \in \text{cl}(X)$ or $r \notin \text{cl}(E(M) - X)$; iff $r \in \text{cl}(X)$ or $r \in \text{cl}^*(X)$; iff $r \in \text{cl}^{(*)}(X)$. \square

Lemma 3.2.8. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid M . Suppose $|X| \geq 3$ and $x \in X$. Then:*

- (i) $x \in \text{cl}^{(*)}(X - \{x\})$; and
- (ii) $(X - \{x\}, Y \cup \{x\})$ is exactly 3-separating if and only if x is in exactly one of $\text{cl}(X - \{x\}) \cap \text{cl}(Y)$ and $\text{cl}^*(X - \{x\}) \cap \text{cl}^*(Y)$.

Proof. Assume that $x \notin \text{cl}^*(X - \{x\})$. Then $x \in \text{cl}(Y)$ by Lemma 2.5.4, so $r(Y) = r(Y \cup \{x\})$. Now suppose that $x \notin \text{cl}(X - \{x\})$. Then $r(X) = r(X - \{x\}) + 1$, and therefore

$$\begin{aligned} r(X) + r(Y) - r(M) &= 2 \\ \implies r(X - \{x\}) + 1 + r(Y \cup \{x\}) - r(M) &= 2 \\ \implies r(X - \{x\}) + r(Y \cup \{x\}) - r(M) &= 1 \end{aligned}$$

But $|X - \{x\}| \geq 2$ so that $(X - \{x\}, Y \cup \{x\})$ is a contradictory 2-separation of M . Therefore $x \notin \text{cl}^*(X - \{x\}) \implies x \in \text{cl}(X - \{x\})$. Thus $x \in \text{cl}^{(*)}(X - \{x\})$; completing the proof of (i). Now, using Lemma 2.5.4, $\lambda(X - \{x\}) = 2$ iff $r(X - \{x\}) + r(Y \cup \{x\}) - r(M) = 2$; iff $r(X - \{x\}) = r(X)$ and $r(Y \cup \{x\}) = r(Y)$, or $r(X - \{x\}) = r(X) - 1$ and $r(Y \cup \{x\}) = r(Y) + 1$; iff $x \in \text{cl}(X - \{x\}) \cap \text{cl}(Y)$, or $x \notin \text{cl}(X)$ and $x \notin \text{cl}(Y)$; iff $x \in \text{cl}(X - \{x\}) \cap \text{cl}(Y)$ or $x \in (\text{cl}^*(X) \cap \text{cl}^*(Y)) - \text{cl}(X)$; iff x is in exactly one of $\text{cl}(X - \{x\}) \cap \text{cl}(Y)$ and $\text{cl}^*(X) \cap \text{cl}^*(Y)$. \square

3.3 Vertical & Cyclic Separations

The following types of k -separations will play a critical role throughout the remainder of this thesis.

Definition 3.3.1. A k -separation $(P, E - P)$ of a matroid $M = (E, \mathcal{I})$ is *vertical* if both $r(P) \geq k$ and $r(E - P) \geq k$. A k -separation $(P, E - P)$ is *cyclic* if both P and $E - P$ contain circuits of M . Let $(X, \{a\}, Y)$ be a partition of E . If $(X \cup \{a\}, Y)$ and $(X, Y \cup \{a\})$ are both vertical 3-separations of M and $a \in \text{cl}(X) \cap \text{cl}(Y)$, then we say that $(X, \{a\}, Y)$ is a *vertical 3-separation* of M . If $(X \cup \{a\}, Y)$ and $(X, Y \cup \{a\})$ are both cyclic 3-separations of M and $a \in \text{cl}^*(X) \cap \text{cl}^*(Y)$, then we say that $(X, \{a\}, Y)$ is a *cyclic 3-separation* of M .

The next two results are used freely throughout.

Lemma 3.3.2. *Let M be a k -connected matroid. A k -separation $(X, E(M) - X)$ of M is a vertical k -separation of M if and only if it is a cyclic k -separation of M^* .*

Proof. Suppose that $(X, E(M) - X)$ is a vertical k -separation of M . Then $\lambda(X) = k - 1$ because M is k -connected. If X is coindependent, then $E(M) - X$ is spanning by Lemma 2.5.2. But then $r(X) = k - 1 + r(M) - r(E(M) - X) = k - 1$ which contradicts the fact that $(X, E(M) - X)$ is vertical. Therefore X must contain a cocircuit. Similarly, $E(M) - X$ must also contain a cocircuit. Thus, both X and $E(M) - X$ contain circuits of M^* and $(X, E(M) - X)$ is a cyclic k -separation of M^* . The converse follows in the same manner. \square

Corollary 3.3.3. *Let M be a k -connected matroid. A partition $(X, \{a\}, Y)$ of M is a vertical k -separation of M if and only if it is a cyclic k -separation of M^* .*

Proof. This follows immediately from Lemma 3.3.2. \square

Lemma 3.3.4. *Let (P, Q) be a cyclic k -separation of a matroid M . If $a \in \text{cl}^*(P) \cap Q$, then $(P \cup \{a\}, Q - \{a\})$ is also a cyclic k -separation of M .*

Proof. Suppose $a \in \text{cl}^*(P) \cap Q$. If $Q - \{a\}$ is independent, then $a \in \text{cl}(Q - \{a\})$ because Q contains a circuit. But this contradicts Lemma 2.5.4. Therefore $Q - \{a\}$ must contain a circuit. \square

Lemma 3.3.5. *Let M be a 3-connected matroid with $c \in E(M)$. If $(P, \{c\}, Q)$ is a cyclic 3-separation of M , then $(\text{cl}^*(P) - \{c\}, \{c\}, Q - \text{cl}^*(P))$ is also a cyclic 3-separation of M .*

Proof. This follows by repeated application of Lemma 3.3.4. \square

Lemma 3.3.6. *Let M be a 3-connected matroid with $b \in E(M)$. If $(X, \{b\}, Y)$ is a vertical 3-separation of M , then $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is also a vertical 3-separation of M .*

Proof. Suppose $e \in \text{cl}(X) \cap Y$. Then $e \in \text{cl}(X \cup \{b\})$, so that by Lemmas 2.5.4 and 3.2.8, $e \in \text{cl}(Y - \{e\})$. Now both of $(X \cup \{b, e\}, Y - \{e\})$ and $(X \cup \{e\}, Y \cup \{b\})$ are exactly 3-separating by Lemma 3.2.8, and it follows that $(X \cup \{e\}, \{b\}, Y - \{e\})$ is a vertical 3-separation of M . By an iteration of this argument, we deduce that $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is also a vertical 3-separation of M . \square

Lemma 3.3.7. *Suppose $(X, \{a\}, Y)$ is a vertical or cyclic 3-separation of a 3-connected matroid M . Then $(\text{cl}(X) \cap \text{cl}^*(X)) - X = \emptyset$.*

Proof. If there exists some $e \in \text{cl}(X) - X$, then $e \notin \text{cl}^*(Y - \{a, e\})$ by Lemma 2.5.4. Therefore $e \in \text{cl}(Y - \{a, e\})$ by Lemma 3.2.8, and so, by Lemma 2.5.4, $e \notin \text{cl}^*(X)$. Therefore $(\text{cl}(X) \cap \text{cl}^*(X)) - X = \emptyset$. \square

Definition 3.3.8. A *parallel class* of a matroid M is a maximal subset of $E(M)$ such that any two members form a parallel pair. A *series class* of M is a parallel class of M^* .

It follows that any 3-connected matroid M does not contain any parallel or series classes of size greater than one, for if it did, then; up to duality, M would contain a parallel class P of size two or greater, giving a contradictory 2-separation $(P, E(M) - P)$ of M . This fact will be used implicitly throughout.

Definition 3.3.9. Let M be a matroid. The *simplification* of M ; denoted $\text{si}(M)$, is the minor of M which is obtained by deleting all loops of M and all but one element from each parallel class of M . The *cosimplification* of M ; denoted $\text{co}(M)$, is defined by $\text{co}(M) = (\text{si}(M^*))^*$.

When deleting and contracting elements from a 3-connected matroid M ; resulting in a minor N of M , it is often necessary to restrict our attention to the matroids $\text{si}(N)$ and $\text{co}(N)$ in order to maintain 3-connectivity. For example, if $r(M) \geq 3$ and $L \subset E(M)$ is a line containing more than two points, then contracting any element $e \in L$ gives a 2-separation $(L - \{e\}, E(M) - L)$ of the matroid M/e because $L - \{e\}$ becomes a parallel class in M/e of size at least two. But if we move to the matroid $\text{si}(M/e)$, this parallel class becomes a single element, which removes the trivial 2-separation that arose upon contraction. This allows us to better examine the global connectivity structure of the matroid M/e , and it may be that we have now found a minor $\text{si}(M/e)$ of M which is indeed 3-connected. If $\text{si}(N)$ is 3-connected, then we say that N is 3-connected *up to parallel classes*. Similarly, if $\text{co}(N)$ is 3-connected, we say that N is 3-connected *up to series classes*.

Lemma 3.3.10. *A 2-separation (X, Y) of a connected matroid M is cyclic if and only if neither X nor Y is a series class.*

Proof. A 2-separation (X, Y) is not cyclic in M iff it is not vertical in M^* ; iff either $r^*(X) = 1$ or $r^*(Y) = 1$; iff X or Y is a parallel class in M^* ; iff X or Y is a series class in M . \square

Lemma 3.3.11. *Let M be a 3-connected matroid. If there exists some $c \in E(M)$ such that $\text{co}(M \setminus c)$ is not 3-connected, then there exists a cyclic 2-separation of $M \setminus c$.*

Proof. Suppose that $\text{co}(M \setminus c)$ is not 3-connected for some $c \in E(M)$. Then there exists a 2-separation (X', Y') of $\text{co}(M \setminus c)$. By Lemma 3.3.10, (X', Y') is cyclic. Therefore (X', Y') is a vertical 2-separation of $(\text{co}(M \setminus c))^* = \text{si}(M^*/c)$. Let $S = E(M^*/c) - E(\text{si}(M^*/c))$. Then by definition of simplification, for all $s \in S$, s is contained in exactly one of $\text{cl}_{M^*/c}(X')$ and $\text{cl}_{M^*/c}(Y')$. Let $S_x = S \cap \text{cl}_{M^*/c}(X')$ and $S_y = S \cap \text{cl}_{M^*/c}(Y')$. Taking $X = X' \cup S_x$ and $Y = Y' \cup S_y$, it then follows directly from the definition of k -separation that $(X \cup S_x, Y \cup S_y)$ is a vertical 2-separation of M^*/c ; hence a cyclic 2-separation of $(M^*/c)^* = M \setminus c$. \square

The following two results are used freely and extensively throughout the remainder of this thesis.

Lemma 3.3.12. *Let M be a 3-connected matroid with $b \in E(M)$. If $\text{si}(M/b)$ is not 3-connected, then there exists a vertical 3-separation $(X, \{b\}, Y)$ of M .*

Proof. Suppose $\text{si}(M/b)$ is not 3-connected. Then there exists a 2-separation (X, Y) of M/b such that $(X \cap E(\text{si}(M/b)), Y \cap E(\text{si}(M/b)))$ is a 2-separation of $\text{si}(M/b)$. We have $r_{M/b}(X) + r_{M/b}(Y) - r(M/b) = 1$, so by Lemma 2.6.5, $r_M(X \cup \{b\}) + r_M(Y \cup \{b\}) - r(M) = 2$. Now if $b \notin \text{cl}(X)$, then $r_M(X) + r_M(Y \cup \{b\}) - r(M) = 1$; which contradicts the fact that M is 3-connected. So $b \in \text{cl}(X)$. Similarly, $b \in \text{cl}(Y)$. As $r_{M/b}(X) \geq 2$, we have $r_M(X) = r_M(X \cup \{b\}) \geq 3$. Similarly, $r_M(Y) \geq 3$. We conclude that $(X, \{b\}, Y)$ is a vertical 3-separation of M . \square

Lemma 3.3.13. *Let M be a 3-connected matroid with $c \in E(M)$. If $\text{co}(M \setminus c)$ is not 3-connected, then there exists a cyclic 3-separation $(P, \{c\}, Q)$ of M .*

Proof. This follows immediately as the dual of Lemma 3.3.12. \square

3.4 Further Results on Connectivity

We conclude this chapter with several results of importance to following chapters.

Lemma 3.4.1. *Let A be a cocircuit of a 3-connected matroid M where $r(A) = 3$ and $e \in A$. If, for some set $\{p, q, r\} \subseteq A - \{e\}$, p, q and r are each contained in distinct parallel classes of M/e , then $\text{si}(M/e)$ is 3-connected.*

Proof. Suppose $\{p, q, r\} \subseteq A - \{e\}$ where p, q and r are each contained in a distinct parallel class of M/e . Assume that $\text{si}(M/e)$ is not 3-connected. Then M has a vertical 3-separation $(X, \{e\}, Y)$. It follows that (X, Y) is a vertical 2-separation of M/e . Without loss of generality, $|X \cap \{p, q, r\}| \geq 2$, so we may assume that $\{p, q\} \subseteq X$. Now, as $r_{M/e}(A - \{e\}) = 2$, we have $r_{M/e}(X \cap (A - \{e\})) = 2$. By the dual of Lemma 3.3.4, it follows that $(\text{cl}_{M/e}(X), Y - \text{cl}_{M/e}(X))$ is a vertical 2-separation of M/e where $A - \{e\} \subseteq \text{cl}_{M/e}(X)$. Then $Y - \text{cl}_{M/e}(X)$ is contained in the hyperplane $E(M) - A$, which implies that $r_M(Y - \text{cl}_{M/e}(X)) = r_{M/e}(Y - \text{cl}_{M/e}(X))$. But then

$$\begin{aligned} \lambda_M(Y - \text{cl}_{M/e}(X)) &= r_M(Y - \text{cl}_{M/e}(X)) + r_M(\text{cl}_{M/e}(X) \cup \{e\}) - r(M) \\ &= r_{M/e}(Y - \text{cl}_{M/e}(X)) + r_{M/e}(\text{cl}_{M/e}(X)) - r(M/e) \\ &= \lambda_{M/e}(Y - \text{cl}_{M/e}(X)) \leq 1 \end{aligned}$$

So that $(\text{cl}_{M/e}(X) \cup \{e\}, Y - \text{cl}_{M/e}(X))$ is a 2-separation of M ; contradicting the fact that M is 3-connected. Therefore $\text{si}(M/e)$ must be 3-connected. \square

The following result; known as Bixby's Lemma [3], shows that up to parallel and series classes, every element of a 3-connected matroid can either be contracted or deleted to give a minor which remains 3-connected.

Lemma 3.4.2. *Let e be an element of a 3-connected matroid M . Then either $\text{si}(M/e)$ or $\text{co}(M \setminus e)$ is 3-connected.*

Proof. Suppose that neither $\text{si}(M/e)$ nor $\text{co}(M \setminus e)$ are 3-connected. Let $(X, \{e\}, Y)$ and $(S, \{e\}, T)$ be vertical and cyclic 3-separations of M respectively.

3.4.2.1. *For all $Z \in \{X \cap S, X \cap T, Y \cap S, Y \cap T\}$, $Z \neq \emptyset$.*

Proof. Suppose $X \cap S = \emptyset$. Then $X \subseteq T$. Therefore $e \in \text{cl}(T) \cap \text{cl}^*(T)$; but this contradicts Lemma 3.3.7. Similarly, each of $X \cap T, Y \cap S$, and $Y \cap T$ are non-empty. \square

3.4.2.2. For all $Z \in \{X \cap S, X \cap T, Y \cap S, Y \cap T\}$, $\lambda(Z) \leq 2$.

Proof. $\lambda(X) = \lambda(S) = 2$. By 3.4.2.1, $|E - (X \cup S)| = |(Y \cap T) \cup \{e\}| \geq 2$, so that by uncrossing, $\lambda(X \cap S) \leq 2$. Similarly, each of $X \cap T, Y \cap S$, and $Y \cap T$ are 3-separating. \square

3.4.2.3. $|X \cap S| \geq 2$ and $|Y \cap T| \geq 2$

Proof. Suppose that $|X \cap S| = 1$. Then $|X \cap T| \geq 2$ and $|Y \cap S| \geq 2$ because $r(X) \geq 3$ and S contains a circuit. So by 3.4.2.2, $\lambda(X \cap T) = 2$ as M is 3-connected. Furthermore, $\lambda(X \cup \{e\}) = \lambda(T \cup \{e\}) = 2$ and $|E - ((X \cup \{e\}) \cup (T \cup \{e\}))| = |Y \cap S| \geq 2$, so that, by uncrossing, $\lambda((X \cup \{e\}) \cap (T \cup \{e\})) = \lambda((X \cap T) \cup \{e\}) = 2$. Now by Lemma 3.2.7, $e \in \text{cl}^{(*)}(X \cap T)$. But $e \in \text{cl}(Y) \cap \text{cl}^*(S)$, so $e \in \text{cl}(Y \cup S) \cap \text{cl}^*(Y \cup S)$; which contradicts Lemma 2.5.4. Therefore $|X \cap S| \neq 1$, and by 3.4.2.2, $|X \cap S| \geq 2$. Similarly, $|Y \cap T| \geq 2$. \square

By 3.4.2.2 and 3.4.2.3, $\lambda(X \cap S) = 2$. Also by 3.4.2.3, $|E - ((X \cup \{e\}) \cup (S \cup \{e\}))| = |Y \cap T| \geq 2$. Therefore, by uncrossing, $\lambda((X \cap S) \cup \{e\}) = 2$. Now by Lemma 3.2.7, $e \in \text{cl}(X \cap S)$ or $e \in \text{cl}^*(X \cap S)$. But then by Lemma 3.3.3, the first possibility contradicts the fact that $e \in \text{cl}^*(T)$; while the second contradicts the fact that $e \in \text{cl}(Y)$. We are left to deduce that at least one of $\text{si}(M/e)$ or $\text{co}(M \setminus e)$ is 3-connected. \square

Lemma 3.4.3. Let $(X, \{b\}, Y)$ be a vertical 3-separation of a 3-connected matroid M . If $z \in X \cap \text{cl}^*(Y)$ then either $(X - \{z\}, \{b\}, Y \cup \{z\})$ is a vertical 3-separation of M or $r((X - \{z\}) \cup \{b\}) = 2$.

Proof. Suppose $z \in X$ is such that $z \in \text{cl}^*(Y)$. By Lemma 3.2.8, $z \in \text{cl}^{(*)}((X \cup \{b\}) - \{z\})$. As $z \in \text{cl}^*(Y)$, $z \notin \text{cl}((X \cup \{b\}) - \{z\})$ by Lemma 2.5.4. So $z \in \text{cl}^*((X \cup \{b\}) - \{z\})$. Therefore $((X \cup \{b\}) - \{z\}, Y \cup \{z\})$ is an exactly 3-separating partition of M by Lemma 3.2.8 (ii). If $|X - \{z\}| = 2$ and $r((X - \{z\}) \cup \{b\}) = 3$, then $r(X) = 3 = r(X \cup \{b\})$, so that $z \in \text{cl}((X - \{z\}) \cup \{b\})$ which; by Lemma 2.5.4, contradicts our assumption that $z \in \text{cl}^*(Y)$. Therefore whenever $|X - \{z\}| = 2$, it must be that $b \in$

$\text{cl}(X - \{z\})$ as required. So assume that $|X - \{z\}| \geq 2$. Now $b \in \text{cl}^{(*)}(X - \{z\})$ by Lemma 3.2.8 (i). As $b \in \text{cl}(Y) \subset \text{cl}(Y \cup \{z\})$, $b \notin \text{cl}^*(X - \{z\})$ by Lemma 2.5.4. So $b \in \text{cl}(X - \{z\})$. Hence, $(X - \{z\}, Y \cup \{b, z\})$ is an exactly 3-separating partition of M by Lemma 3.2.8 (ii). Therefore, if $r(X - \{z\}) \geq 3$, then $(X - \{z\}, \{b\}, Y \cup \{z\})$ is a vertical 3-separation of M . Otherwise, $r(X - \{z\}) = r((X - \{z\}) \cup \{b\}) = 2$. \square

Theorem 3.4.4. *Let M be a 3-connected matroid with $r(M) \geq 4$. Suppose that C^* is a rank 3 cocircuit of M . If there exists some $c \in C^*$ such that $r(C^* - \{c\}) = 3$, then $\text{co}(M \setminus c)$ is 3-connected.*

Proof. Suppose that $c \in C^*$ and $r(C^* - \{c\}) = 3$, but $\text{co}(M \setminus c)$ is not 3-connected. Then by Lemma 3.3.11, there exists a cyclic 2-separation (P, Q) of $M \setminus c$.

As C^* is a cocircuit, we have $r(E(M) - C^*) = r(M) - 1$. Hence $\lambda_M(C^*) = 2$. Noting that $r(E(M) - C^*) \geq 3$ because $r(M) \geq 4$, we have that $(C^*, E(M) - C^*)$ is a vertical 3-separation of M . For clarity, let $X = C^* - \{c\}$, and $Y = E(M) - C^*$ so that $(X \cup \{c\}, Y)$ is a 3-separation of M .

3.4.4.1. *Each of the intersections $X \cap P, X \cap Q, Y \cap P$, and $Y \cap Q$ are non-empty.*

Proof. If $X \cap P = \emptyset$, then $X \subseteq Q$. But then $c \in \text{cl}(X) \subseteq \text{cl}(Q)$ so that $(P, Q \cup \{c\})$ is a contradictory 2-separation of M . Similarly, $X \cap Q \neq \emptyset$. If $Y \cap P = \emptyset$, then $Y \subseteq Q$. Now Y is a hyperplane of M and $X \cap Q \neq \emptyset$, so that $r(Q) = r(M)$ and $c \in \text{cl}(Q)$; again giving a contradictory 2-separation $(P, Q \cup \{c\})$ of M . Similarly, $Y \cap Q \neq \emptyset$. \square

3.4.4.2. *For all $Z \in \{X \cap P, X \cap Q, Y \cap P, Y \cap Q\}$, $\lambda_{M \setminus c}(Z) = \lambda_M(Z)$.*

Proof. By hypothesis, $r(M \setminus c) = r(M)$. If $Z \in \{X \cap P, X \cap Q\}$, then by 3.4.4.1 and the fact that Y is a hyperplane, $c \in \text{cl}(E(M) - Z)$. If $Z \in \{Y \cap P, Y \cap Q\}$, then $c \in \text{cl}(X) \subseteq \text{cl}(E(M) - Z)$, and in either case we have $\lambda_{M \setminus c}(Z) = \lambda_M(Z)$. \square

Now, by the submodularity of the connectivity function λ , we see that $\lambda_{M \setminus c}(X \cap P) + \lambda_{M \setminus c}(X \cup P) \leq \lambda_{M \setminus c}(X) + \lambda_{M \setminus c}(P) = 2 + 1 = 3$. So, $\lambda_{M \setminus c}(X \cap P) + \lambda_{M \setminus c}(Y \cap Q) \leq \lambda_{M \setminus c}(X) + \lambda_{M \setminus c}(P) = 3$. By 3.4.4.2, $\lambda_M(X \cap P) + \lambda_M(Y \cap Q) \leq 3$. So at least one of $\{X \cap P, Y \cap Q\}$ is 2-separating. As M

is 3-connected, it then must be the case that at least one of $\{X \cap P, Y \cap Q\}$ has cardinality equal to 1. Now, applying submodularity of the function λ and 3.4.4.2 again to $X \cap Q$, we deduce that at least one of $\{X \cap Q, Y \cap P\}$ has cardinality equal to 1. We cannot have $|X \cap P| = 1 = |X \cap Q|$ as $|X| \geq 3$. It also cannot be that $|Y \cap P| = 1 = |Y \cap Q|$ as $r(Y) \geq 3$. We are left to conclude that $|P| = 2$ or $|Q| = 2$. But (P, Q) is a cyclic 2-separation of $M \setminus c$, so either P or Q must be a parallel pair of $M \setminus c$; which implies the existence of a contradictory parallel pair in the 3-connected matroid M . Thus $\text{co}(M \setminus c)$ must be 3-connected. \square

Corollary 3.4.5. *Let M be a 3-connected matroid with a rank 3 cocircuit C^* such that $|C^*| \geq 4$ and $r(M) \geq 4$. Let B be a basis of M . Then there exists some $c \in C^* \cap (E(M) - B)$ such that $\text{co}(M \setminus c)$ is 3-connected.*

Proof. If $r(C^* - \{z\}) = 3$ for all $z \in C^*$, the result follows immediately from Theorem 3.4.4 and the fact that $|C^*| > r(C^*)$. So assume that $r(C^* - \{d\}) = 2$ for some $d \in C^*$. Now take any $c \in (C^* - \{d\}) \cap (E(M) - B) \neq \emptyset$. Then $r(C^* - \{c, d\}) = 2$ because M is 3-connected and it follows that $r(C^* - \{c\}) = 3$. Applying Theorem 3.4.4, we get that $\text{co}(M \setminus c)$ is 3-connected. \square

Definition 3.4.6. Let M be a 3-connected matroid, and B a basis of M . Suppose that for some $b_1 \in B$, $\text{si}(M/b_1)$ is not 3-connected. If $(X, \{b_1\}, Y)$ is a vertical 3-separation of M ; where $Y = \{x_1, x_2, b_2, c\}$ is a cocircuit with $Y \cap B = \{b_2\}$, $\text{si}(M/b_2)$ is not 3-connected, and both $\{x_1, x_2, b_2\}$ and $\{b_1, c, b_2\}$ are triangles, then we shall call Y a *near-fan* of M .

Corollary 3.4.7. *Let M be a 3-connected matroid and B a basis of M . If $Y = \{x_1, x_2, b_2, c\} \subset E(M)$ is a near-fan of M , then both $\text{co}(M \setminus x_1)$ and $\text{co}(M \setminus x_2)$ are 3-connected.*

Proof. This is an immediate consequence of Theorem 3.4.4. \square

Theorem 3.4.8. *Let M be a 3-connected matroid with a triad $\{a, b, c\}$ and a circuit $\{a, b, c, d\}$. Then at least one of the following holds.*

- (i) *Either $\text{co}(M \setminus a)$ or $\text{co}(M \setminus c)$ is 3-connected.*
- (ii) *There exists $a', c' \in E(M)$ such that both $\{a, a', b\}$ and $\{c, c', b\}$ are triangles.*

(iii) There exists $r \in E(M)$ such that $\{a, b, c, r\}$ is a coline.

Proof. We shall assume that neither (i) nor (ii) holds and prove that in this case, (iii) must hold. Now, as neither $\text{co}(M \setminus a)$ nor $\text{co}(M \setminus c)$ are 3-connected, it follows from Lemma 3.3.11 that there are cyclic 2-separations (P, Q) and (V, W) of $M \setminus a$ and $M \setminus c$ respectively.

Consider (P, Q) . We can assume without loss of generality that $d \in Q$. If $\{b, c\} \subseteq Q$, then $(P, Q \cup \{a\})$ is a 2-separation of M as $a \in \text{cl}(\{b, c, d\})$. So assume $b \in P$. Then $c \in \text{cl}_{M \setminus a}^*(P)$ by Lemma 2.5.4, and it follows from Lemma 3.3.4 that $(P \cup \{c\}, Q - \{c\})$ is a cyclic 2-separation of $M \setminus a$. Thus, relabelling for clarity, we have a cyclic 2-separation $(X \cup \{b, c\}, Y)$ of $M \setminus a$ where $d \in Y$. Similarly, we obtain a cyclic 2-separation $(S \cup \{b, a\}, T)$ of $M \setminus c$ where $d \in T$.

3.4.8.1. *If $\{b, c\}$ or $\{b, a\}$ is contained in a triangle, then the theorem holds.*

Proof. Let $\{b, c, c'\}$ be a triangle. By our original assumption, $\{b, a\}$ is not contained in a triangle. Consider the 2-separation $(S \cup \{b, a\}, T)$ of $M \setminus c$. As $S \cup \{a, b\}$ contains a circuit and $\{b, a\}$ is not in a triangle, we have $|S| \geq 2$. If $c' \in S$, then $c \in \text{cl}_M(S \cup \{b, a\})$, so that $(S \cup \{a, b, c\}, T)$ is a 2-separation of M . Thus $c' \in T$. Now with the use of Lemma 2.6.5 we have

$$\begin{aligned} \lambda_{M \setminus c/b}(T) &= r_{M \setminus c/b}(T) + r_{M \setminus c/b}(S \cup \{a\}) - r(M \setminus c/b) \\ &= r_{M \setminus c}(T \cup \{b\}) - 1 + r_{M \setminus c}(S \cup \{a, b\}) - 1 - (r(M \setminus c) - 1) \\ &= r_{M \setminus c}(T) + r_{M \setminus c}(S \cup \{a, b\}) - r(M) \\ &= r(M \setminus c) + 1 - r(M) = 1 \end{aligned}$$

Note that $r(\{a, b, c, c', d\}) = 3$ because $\{a, b, c, d\}$ is a rank 3 circuit and $\{b, c, c'\}$ is a triangle. So $r_{M \setminus c}(\{a, b, c', d\}) = 3$. Therefore,

$$r_{M \setminus c/b}(\{a, c', d\}) = r_{M \setminus c}(\{a, b, c', d\}) - 1 = 2.$$

We also have $r_{M \setminus c/b}(\{c', d\}) = r_{M \setminus c}(\{b, c', d\}) - 1 = 2$. Hence $a \in \text{cl}_{M \setminus c/b}(\{c', d\}) \subseteq \text{cl}_{M \setminus c/b}(T)$. Therefore it must be that $\lambda_{M \setminus c/b}(T \cup \{a\}) = 1$. So $r_{M \setminus c/b}(T \cup \{a\}) + r_{M \setminus c/b}(S) = r(M \setminus c/b) + 1$ and

$$\begin{aligned} r_{M \setminus c/b}(T \cup \{a\}) + r_{M \setminus c/b}(S) &= r(M \setminus c/b) + 1 \\ \implies r_{M \setminus c}(T \cup \{a, b\}) + r_{M \setminus c/b}(S) &= r(M \setminus c/b) + 2 \end{aligned}$$

$$\begin{aligned} \implies r_{M \setminus c}(T \cup \{a, b\}) + r_{M \setminus c/b}(S) &= r(M) + 1 \\ \implies r_{M \setminus c}(T \cup \{a, b\}) + r_{M \setminus c}(S) &= r(M) + 1 \end{aligned}$$

So that $\lambda_{M \setminus c}(T \cup \{a, b\}) = 1$. But then $c \in \text{cl}(\{a, b, d\})$ implies that $\lambda_M(T \cup \{a, b, c\}) = 1$, contradicting the fact that M is 3-connected. Therefore $\{b, c\}$ can not be contained in a triangle. Similarly for $\{b, a\}$. \square

Thus we may now assume that neither $\{b, a\}$ nor $\{b, c\}$ is contained in a triangle. From this and the fact that both $X \cup \{b, c\}$ and $S \cup \{b, a\}$ contain circuits, we deduce that $|X| \geq 2$ and $|S| \geq 2$. In what follows, let $M' = M \setminus \{a, b, c\}$.

3.4.8.2. $c \in \text{cl}(X \cup \{b\})$ and $a \in \text{cl}(S \cup \{b\})$.

Proof. Suppose that $c \notin \text{cl}(X \cup \{b\})$. Then $r(X) = r(X \cup \{b, c\}) - 2$. But $r(M') = r(M) - 1$. We have

$$\begin{aligned} r(X \cup \{b, c\}) + r(Y) &\leq r(M) + 1 \\ \implies r(X) + 2 + r(Y) &\leq r(M') + 2 \\ \implies r(X) + r(Y) &\leq r(M') \end{aligned}$$

So that (X, Y) is a separation of M' . But $r(Y \cup \{b, c\}) \leq r(Y) + 2$, and $a \in \text{cl}_M(Y \cup \{b, c\})$, so that $r(Y \cup \{a, b, c\}) \leq r(Y) + 2$. Now

$$\begin{aligned} r(X) + r(Y \cup \{a, b, c\}) &\leq r(X) + r(Y) + 2 \\ &\leq r(M') + 2 \\ &\leq r(M) + 1. \end{aligned}$$

This gives $(X, Y \cup \{a, b, c\})$ as a 2-separating partition of M ; which is contradictory as $|X| \geq 2$. Similarly $a \in \text{cl}(S \cup \{b\})$. \square

3.4.8.3. $\lambda_{M'}(X) = \lambda_{M'}(Y) = \lambda_{M'}(S) = \lambda_{M'}(T) = 1$.

Proof. Using 3.4.8.2,

$$\begin{aligned}
\lambda_{M'}(X) &= r_{M'}(X) + r_{M'}(Y) - r(M') \\
&= r_{M'}(X) + r_{M'}(Y) - r(M) + 1 \\
&= r_{M \setminus a}(X) + r_{M \setminus a}(Y) - r(M) + 1 \\
&= r_{M \setminus a}(X \cup \{b\}) - 1 + r_{M \setminus a}(Y) - r(M) + 1 \\
&= r_{M \setminus a}(X \cup \{b, c\}) + r_{M \setminus a}(Y) - r(M) \\
&= r(M) + 1 - r(M) \\
&= 1 = \lambda_{M'}(Y)
\end{aligned}$$

The result that $\lambda_{M'}(S) = \lambda_{M'}(T) = 1$ follows similarly. \square

3.4.8.4. *If $Z \subseteq X$ or $Z \subseteq S$, then $\lambda_M(Z) \leq \lambda_{M'}(Z) + 1$.*

Proof. Suppose $Z \subseteq X$. Let $Z' = E(M') - Z$. Note that $d \in Z'$ so $r(Z' \cup \{a, b, c\}) \leq r(Z') + 2$. Then $r(M) = r(M') + 1$ implies that

$$\begin{aligned}
\lambda_M(Z) &= r(Z) + r(Z' \cup \{a, b, c\}) - r(M) \\
&= r_{M'}(Z) + r(Z' \cup \{a, b, c\}) - r(M') - 1 \\
&\leq r_{M'}(Z) + r(Z') + 2 - r(M') - 1 \\
&\leq r_{M'}(Z) + r_{M'}(Z') - r(M') + 1 \\
&\leq \lambda_{M'}(Z) + 1
\end{aligned}$$

Similarly if $Z \subseteq S$. \square

3.4.8.5. *If $Z \subseteq Y - \{d\}$ or $Z \subseteq T - \{d\}$, then $\lambda_M(Z) = \lambda_{M'}(Z)$.*

Proof. Suppose $Z \subseteq Y - \{d\}$. Let $Z' = E(M') - Z$. Note that $d \in Z'$ and $X \subset Z'$. Now $r(Z' \cup \{b\}) = r(Z') + 1$, but by 3.4.8.2, $c \in \text{cl}(X \cup \{b\})$, so $c \in \text{cl}(Z' \cup \{b\})$. Hence, $r(Z' \cup \{b, c\}) = r(Z') + 1$. But then $a \in \text{cl}(Z' \cup \{b, c\})$, so $r(Z' \cup \{a, b, c\}) = r(Z') + 1$. Therefore

$$\begin{aligned}
\lambda_M(Z) &= r(Z) + r(Z' \cup \{a, b, c\}) - r(M) \\
&= r(Z) + r(Z') + 1 - r(M) \\
&= r(Z) + r(Z') - r(M') = \lambda_{M'}(Z)
\end{aligned}$$

as required. Similarly for any $Z \subseteq T - \{d\}$. \square

3.4.8.6. $T \cap X \neq \emptyset$, and $S \cap Y \neq \emptyset$.

Proof. If $T \cap X = \emptyset$, then $X \subseteq S$, and by 3.4.8.2, $\{a, c\} \subseteq \text{cl}(S \cup \{b\})$. We once again obtain a contradictory 2-separation $(S \cup \{a, b, c\}, T)$ of M . Similarly for $S \cap Y$. \square

Applying submodularity of the connectivity function together with 3.4.8.3 we have

$$\begin{aligned} \lambda_{M'}(T \cup X) + \lambda_{M'}(T \cap X) &\leq \lambda_{M'}(T) + \lambda_{M'}(X) \\ \implies \lambda_{M'}(S \cap Y) + \lambda_{M'}(T \cap X) &\leq 1 + 1 = 2 \end{aligned}$$

But by 3.4.8.5, $\lambda_{M'}(S \cap Y) = \lambda_M(S \cap Y)$ and $\lambda_{M'}(T \cap X) = \lambda_M(T \cap X)$. This result combined with 3.4.8.6 and the fact that M is 3-connected implies that we must have $|S \cap Y| = |T \cap X| = 1$. We also know that $|Y| \geq 3$ because $(X \cup \{b, c\}, Y)$ is cyclic. Thus $|Y \cap T| \geq 2$.

3.4.8.7. $\lambda_{M'}(Y \cap T) \geq 2$.

Proof. By 3.4.8.2, $\{a, c\} \subseteq \text{cl}(X \cup S \cup \{b\})$, so as $r(X \cup S \cup \{b\}) = r(X \cup S) + 1$, we have $r(X \cup S \cup \{a, b, c\}) = r(X \cup S) + 1$. Hence $\lambda_M(Y \cap T, X \cup S \cup \{a, b, c\}) = \lambda_{M'}(Y \cap T, X \cup S)$. Thus, if $\lambda_{M'}(Y \cap T) \in \{0, 1\}$, we contradict the 3-connectivity of M . \square

Now $\lambda_{M'}(S \cap X) + \lambda_{M'}(T \cap Y) \leq \lambda_{M'}(S) + \lambda_{M'}(X) \leq 1 + 1 = 2$. So by 3.4.8.7, $\lambda_{M'}(S \cap X) = 0$. By 3.4.8.4, we deduce that $|S \cap X| = 1$. Let $S \cap X = \{r\}$. Let $\{a, b, c\} = A$. As $\lambda_{M'}(S \cap X) = 0$, we have $\lambda_{M'}(Y \cup T) = 0$. Therefore $r_M(Y \cup T) = r_{M'}(Y \cup T) = r(M') - 1 = r(M) - 2$. Thus $\lambda_M(Y \cup T) = r(M) - 2 + r(A \cup \{r\}) - r(M) = r(A \cup \{r\}) - 2 \leq 2$, so that $(Y \cup T, A \cup \{r\})$ must be a 3-separation of M . Now by Lemma 3.2.7, $r \in \text{cl}^{(*)}(A)$. If $r \in \text{cl}(A)$, then $(S \cup \{a, b, c\}, T)$ is a contradictory 2-separation of M . We conclude that $r \in \text{cl}^*(A)$ and that $\{a, b, c, r\}$ is a coline, giving (iii). \square

3.5 Fixed Basis Connectivity

Given a 3-connected matroid M with $e \in E(M)$, Bixby's Lemma (see 3.4.2) tells us that either $\text{si}(M/e)$ or $\text{co}(M \setminus e)$ is 3-connected. However, situations can arise where we specifically want to contract certain elements, or delete certain elements, all while maintaining 3-connectivity. In such circumstances, Bixby's Lemma may be of limited use. This section introduces one important example of such a situation.

Suppose we have an \mathbb{F} -representable matroid M . As M is invariant under performing standard row operations on a representation of M , it is natural that we want to deal with representations of the form $[I_r, D]$ where I_r is an $r \times r$ identity matrix and D is an $r \times (|E(M)| - r)$ matrix. Such a matrix representation is said to be in *standard form*. Now if $A = [I_r, D]$ is a representation of M over \mathbb{F} in standard form, then the set of column vectors contained in I_r form a basis for the space \mathbb{F}^r . As the rank of a matroid corresponds to the dimension of the rowspace in a representation of that matroid, it then follows that $r(M) = r$, and therefore, the set of column labels for columns in I_r constitute a basis B of the matroid M . Hence from our natural desire to work with representations that are in standard form, we are in effect choosing a particular basis for M . Any information present in the submatrix D is observed *relative to the basis B* . Clearly, a matroid will have many possible standard form representations. For example, if d is in some basis B_d of M , and d corresponds to some column of D , then (see Lemma 2.1.7), for some $i \in \{1, \dots, r\}$, we may pivot on the i -th entry of d — transforming it into the i -th unit vector — and interchange columns d and i . The resulting matrix is another standard form representation $A' = [I_r, G]$ of M ; corresponding to the basis $B' = (B - \{i\}) \cup \{d\}$ of M . However in general, the entries in the submatrix G will differ significantly from the entries of the submatrix D . Thus, important information that may be present in D can be lost when switching to the representation A' relative to the new basis B' .

We observed at the end of Section 2.6 that an \mathbb{F} -representation for any minor N of an \mathbb{F} -representable matroid M can be found by taking an \mathbb{F} -representation A of M and performing a sequence of operations consisting of pivots, row deletions and column deletions. We will examine what happens when our starting representation A of M is in standard form. Suppose that $[I_r, D]$ is a representation — in standard form — of M . Then the columns of I_r correspond to a basis B of M , and therefore, the columns of D correspond

to a cobasis $E(M) - B$ of M .

Suppose that $d \in E(M) - B$. Then d corresponds to some column of the submatrix D . We will assume that d is not a loop of M to avoid the trivial case. Consider the matroid M/d . Recall that the first step in obtaining a representation of M/d is to pivot on some non-zero i -th entry of column d ; transforming it into the i -th unit vector. If we then switch columns d and i before completing our contraction of d , we find that — upon contracting d — we obtain a representation of M/d which is in standard form. But, as observed above, such a procedure corresponds to changing the basis with which we are observing the representation with respect to, and carrying out such a process means changing potentially important information present in the submatrix D . Now consider the matroid $M \setminus d$. To obtain a representation for $M \setminus d$, we simply delete column d of D . Doing so gives a representation — in standard form — of $M \setminus d$. Moreover, in this case, all remaining entries in the submatrix D are completely unchanged.

Now suppose that $b \in B$. Then b corresponds to some column of the identity matrix I_r . Consider the matroid $M \setminus b$. We obtain a representation of this matroid by deleting column b , but doing so leaves us with a representation which is not in standard form. To then find a standard form representation of $M \setminus b$, we need to pivot on a non-zero entry of some column of D to obtain another unit vector. But doing so will again often perturb information present in entries of D which we may be wanting to maintain. On the other hand, consider the matroid M/b . As b already represents some i -th unit vector, contracting b simply corresponds to removing row i and column b . This gives us a representation of M/b which is in standard form, where all entries of D that remain are unaltered.

So suppose we have a representation $[I_r, D]$ — in standard form — of a matroid M over \mathbb{F} . As discussed, this corresponds to a particular basis B and cobasis $E(M) - B$ of M . Now suppose that one wished to study a representation $[I_q, D']$ of a minor N of M over \mathbb{F} , and one required that relevant information present in D remain observable in D' . As we've seen, such a requirement can be met as long as we restrict ourselves to the situation where N is some minor obtained from M by contracting elements from the basis B and/or deleting elements of the cobasis $E(M) - B$. If N is a minor of M found in such a manner, then a representation $[I_q, D']$ — in standard form — of N is easily found, and entries of D' correspond identically to entries present in the submatrix D of our original representation.

For the remainder of this thesis, we are interested in what can be done in such a situation. From a matroidal point of view, we have some matroid M , and some particular basis B of M , and we want to study minors of M that are found by contracting elements of B and deleting elements of $E(M) - B$. But of course, even for minors found in this way, if the representations themselves are to be of any use as an inductive tool, we need some sense of uniqueness of representations; and for that, we require 3-connectivity.

Oxley et al [10] studied this problem and found that: given a matroid M and basis B of M ; provided M has no 4-element fans, there is always either an element $b \in B$ such M/b is 3-connected, or an element $c \in E(M) - B$ such that $M \setminus c$ is 3-connected. In this work, we are predominantly concerned with what can be done when wanting to maintain 3-connectivity up to parallel and series classes. We make the following definition:

Definition 3.5.1. Let M be a 3-connected matroid, B a basis of M , and $e \in E(M)$. We say that e is *removable with respect to B* if either:

- (i) $e \in B$ and $\text{si}(M/e)$ is 3-connected, or
- (ii) $e \notin B$ and $\text{co}(M \setminus e)$ is 3-connected.

The following two lemmas show that the property of being removable with respect to a basis behaves nicely under duality.

Lemma 3.5.2. *Let M be a 3-connected matroid, and let B be a basis of M . Then $e \in E(M)$ is removable with respect to B if and only if e is removable with respect to the basis $E(M) - B$ in M^* .*

Proof. $e \in E(M)$ is removable with respect to B iff $e \in B$ and $\text{si}(M/e)$ is 3-connected, or $e \notin B$ and $\text{co}(M \setminus e)$ is 3-connected; iff $e \notin E(M) - B$ and $(\text{si}(M/e))^*$ is 3-connected, or $e \in E(M) - B$ and $(\text{co}(M \setminus e))^*$ is 3-connected; iff $e \notin E(M) - B$ and $\text{co}(M^* \setminus e)$ is 3-connected, or $e \in E(M) - B$ and $\text{si}(M^*/e)$ is 3-connected; iff e is removable with respect to the basis $E(M) - B$ in M^* . \square

Lemma 3.5.3. *Let M be a 3-connected matroid, and let B be a basis of M . Let K denote the set of elements which are removable with respect to B . Let K^* denote the set of elements which are removable with respect to the basis $E(M) - B$ in M^* . Then $K = K^*$.*

Proof. This follows immediately from Lemma 3.5.2. \square

In proving results on 3-connectivity in matroid theory, it is often advantageous to exclude certain substructures from the matroids we are considering. Fans are an example of such a substructure.

Definition 3.5.4. Let M be a matroid. A set $F \subseteq E(M)$ is a *fan* if there is an ordering (f_1, \dots, f_n) of the elements of F such that, for all $i \in \{1, \dots, n-2\}$,

- (i) $\{f_i, f_{i+1}, f_{i+2}\}$ is either a triangle or a triad, and
- (ii) when $\{f_i, f_{i+1}, f_{i+2}\}$ is a triangle, $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triad, and when $\{f_i, f_{i+1}, f_{i+2}\}$ is a triad, $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triangle.

The focus of the work following will be on matroids that do not contain any 4-element fans, that is: fans of cardinality four. Excluding matroids that do contain such structures is not greatly confining for our purposes. For example, it can readily be shown that any excluded minor for $GF(q)$ does not contain any 4-element fans. The following lemma notes that fans are also well behaved under duality.

Lemma 3.5.5. *Let M be a matroid. Then F is a fan of M if and only if it is a fan of M^* . In particular, if M does not contain any 4-element fans, then neither does M^* .*

Proof. This follows directly from the definition of a fan and the fact that triangles and triads are dual to one another. \square

Although not explicitly stated therein, it can be deduced directly from results in the paper by Oxley et al [10] that for a matroid M with no 4-element fans, and a basis B of M , there always exists two distinct elements which are removable with respect to B . In the next chapter, we look to strengthen this result.

Chapter 4

The Existence of Removable Elements

In this chapter we prove the following result:

Theorem 4.0.1. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M . Let K denote the set of elements which are removable with respect to B . If $|E(M)| \geq 4$, then $|K| \geq 4$.*

4.1 Preliminary Results

Theorem 4.1.1. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M such that for some $b \in B$, $\text{si}(M/b)$ is not 3-connected. Let $(X, \{b\}, Y)$ be a vertical 3-separation of M with $Y \cup \{b\}$ closed. Then either:*

- (i) *there exists $b_1, b_2 \in X \cap B$ with $b_1 \neq b_2$ such that both $\text{si}(M/b_1)$ and $\text{si}(M/b_2)$ are 3-connected, or*
- (ii) *there exists $b_1 \in X \cap B, d_1 \in X \cap (E(M) - B)$ such that both $\text{si}(M/b_1)$ and $\text{co}(M \setminus d_1)$ are 3-connected, or*
- (iii) *there exists $d_1, d_2 \in X \cap (E(M) - B)$ with $d_1 \neq d_2$ such that both $\text{co}(M \setminus d_1)$ and $\text{co}(M \setminus d_2)$ are 3-connected, or*

(iv) $X \cap B = \{b_c\}$, $\text{si}(M/b_c)$ is 3-connected, $X = \{a, b_c, c\}$, $X \cup \{b\}$ is a circuit, and neither $\text{co}(M \setminus a)$ nor $\text{co}(M \setminus c)$ is 3-connected.

Proof. We shall assume that (i) does not hold and show that either (ii), (iii) or (iv) does. The proof is separated into 2 cases according the possible structure of X .

Case (i): X contains no element b_x such that $\text{si}(M/b_x)$ is 3-connected.

Assume as stated that for all $b_x \in B \cap X$, $\text{si}(M/b_x)$ is not 3-connected. We shall show that in this case, (iii) must hold.

As $b \in \text{cl}(X)$ and $r(X) \geq 3$, we have $r(Y) < r(M)$, so that $X \cap B \neq \emptyset$. Note that for each $b_x \in B \cap X$, there exists a vertical 3-separation $(S_{b_x}, \{b_x\}, T_{b_x})$ of M . If, for some $b_x \in X \cap B$, there is a vertical 3-separation $(S, \{b_x\}, T)$ of M such that S or T is contained in $X \cup \{b\}$, then by Lemma 3.3.6 and a possible relabelling, there exists such a vertical 3-separation where $S \subseteq X \cup \{b\}$ and $T \cup \{b_x\}$ is closed. Then $S \subseteq (X - \{b_x\}) \cup \{b\}$. If equality holds here, then $T = Y$. But $b_x \in \text{cl}(T)$ so $b_x \in \text{cl}(Y)$; a contradiction. Hence $S \subset (X - \{b_x\}) \cup \{b\}$. Now relabel so that $(S, \{b_x\}, T)$ becomes $(X, \{b\}, Y)$. By an iteration of this procedure, we eventually obtain a vertical 3-separation $(X, \{b\}, Y)$ of M with $Y \cup \{b\}$ closed such that if $(S, \{b_x\}, T)$ is a vertical 3-separation of M with $b_x \in X \cap B$, then neither S nor T is contained in $X \cup \{b\}$. Moreover, we maintain the property that there is no element $b_x \in X \cap B$ such that $\text{si}(M/b_x)$ is 3-connected.

Let b_x be an element of $X \cap B$, and let $(P, \{b_x\}, Q)$ be a vertical 3-separation of M . Without loss of generality, we may assume that $b \in Q$. Moreover, we may assume by Lemma 3.3.6 that $Q \cup \{b_x\}$ is closed.

4.1.1.1. *Each of the intersections $X \cap P, X \cap Q, Y \cap P$, and $Y \cap Q$ is non-empty.*

Proof. If $X \cap P$ or $X \cap Q$ is empty, then P or Q is contained in $Y \cup \{b\}$, and so $b_x \in \text{cl}(Y \cup \{b\})$; which contradicts the fact that $Y \cup \{b\}$ is closed. Thus $X \cap P$ and $X \cap Q$ are non-empty. If $Y \cap P$ or $Y \cap Q$ is empty, then P or Q is contained in $X \cup \{b\}$; a contradiction. \square

4.1.1.2. $\lambda(X \cap P) \leq 2$.

Proof. As $E - (X \cup P) = (Y \cap Q) \cup \{b\}$, we have $|E - (X \cup P)| \geq 2$. Thus, since $\lambda(X) = 2 = \lambda(P)$, it follows by uncrossing that $\lambda(X \cap P) \leq 2$. \square

4.1.1.3. $r((X \cap P) \cup \{b_x\}) = 2$

Proof. If $|X \cap P| = 1$, this follows immediately as M is 3-connected. Since $X \cap P$ is non-empty, we may now assume that $|X \cap P| \geq 2$. We have $\lambda(X) = 2 = \lambda(P \cup \{b_x\})$, and $|E - (X \cup (P \cup \{b_x\}))| \geq 2$, so that, by uncrossing, $\lambda(X \cap (P \cup \{b_x\})) \leq 2$. With 4.1.1.2, we deduce that $\lambda(X \cap P) = \lambda((X \cap P) \cup \{b_x\}) = 2$. By Lemma 3.2.7, $b_x \in \text{cl}^{(*)}(X \cap P)$. If $b_x \in \text{cl}^*(X \cap P)$, then by Lemma 2.5.4, $b_x \notin \text{cl}(Y \cup Q)$; a contradiction. So $b_x \in \text{cl}(X \cap P)$. If $r((X \cap P) \cup \{b_x\}) \geq 3$, then $(X \cap P, \{b_x\}, Y \cup Q)$ is a vertical 3-separation of M . But this contradicts our construction of $(X, \{b\}, Y)$. So $r((X \cap P) \cup \{b_x\}) = 2$. \square

Suppose $|Y \cap P| = 1$. Then $|X \cap P| \geq 2$. If $|X \cap P| = 2$, then, as $Q \cup \{b_x\}$ is closed, P is a triad. As $(X \cap P) \cup \{b_x\}$ is a triangle, $P \cup \{b_x\}$ is a 4-element fan: a contradiction. Thus we may assume that $|X \cap P| \geq 3$. Then $(X \cap P) \cup \{b_x\}$ is a rank-2 set having at least four elements. By Lemmas 3.2.2 and 2.3.3, we obtain distinct $d_1, d_2 \in (E - B) \cap X$ such that both $M \setminus d_1$ and $M \setminus d_2$ are 3-connected, so (iii) holds. So assume that $|Y \cap P| \geq 2$.

4.1.1.4. $r((X \cap Q) \cup \{b, b_x\}) = 2$.

Proof. Since $\lambda(X \cup \{b\}) = 2 = \lambda(Q \cup \{b_x\})$ and $|E - ((X \cup \{b\}) \cup (Q \cup \{b_x\}))| = |Y \cap P| \geq 2$, it follows by uncrossing that $\lambda((X \cap Q) \cup \{b, b_x\}) \leq 2$. But $|(X \cap Q) \cup \{b, b_x\}| \geq 3$ and so $\lambda((X \cap Q) \cup \{b, b_x\}) = 2$. Noting that $P \subseteq E - ((X \cap Q) \cup \{b, b_x\})$, we have $b_x \in \text{cl}(E - ((X \cap Q) \cup \{b, b_x\}))$, and it follows from Lemmas 3.2.7 and 3.2.8 that $b_x \in \text{cl}((X \cup \{b\}) \cap Q)$. If $r((X \cap Q) \cup \{b, b_x\}) \geq 3$, then it follows that $((X \cup \{b\}) \cap Q, \{b_x\}, E - ((X \cap Q) \cup \{b, b_x\}))$ is a vertical 3-separation that contradicts the choice of b . Therefore, $r((X \cap Q) \cup \{b, b_x\}) \leq 2$ and 4.1.1.4 holds. \square

Now let $L_1 = \{b\} \cup (X \cap Q)$ and $L_2 = \{b_x\} \cup (X \cap P)$. We have $|L_1| \geq 2$ and $|L_2| \geq 2$, and both $\text{cl}(L_1)$ and $\text{cl}(L_2)$ are lines. By 4.1.1.1, $|\text{cl}(L_1)| \geq 3$. If $|\text{cl}(L_1)| \geq 4$, then by Lemmas 3.2.2 and 2.3.3 we again obtain distinct $d_1, d_2 \in (E - B) \cap X$ such that (iii) holds. So we may assume that $|\text{cl}(L_1)| = 3$. Let $\text{cl}(L_1) - \{b, b_x\} = \{a\}$. Suppose that $(L_2 - \{b_x\}) \cap B = \{b'\}$. Then, in M/b' , each of b, b_x , and a are contained in some distinct parallel class by an

application of Lemma 2.6.5. Thus, by Lemma 3.4.1, it follows that $\text{si}(M/b')$ is 3-connected: a contradiction. We conclude that $(L_2 - \{b_x\}) \cap B = \emptyset$. If $|\text{cl}(L_2)| \geq 4$, then $|L_2 \cap (E - B)| \geq 2$ by Lemma 2.3.3, so by Lemma 3.2.2, (iii) holds. Thus we may assume that $|\text{cl}(L_2)| \in \{2, 3\}$. Since M has no 4-element fans, and $Y \cup \{b\}$ is closed it must be that $|L_2 - \{b_x\}| = 2$. Let $L_2 - \{b_x\} = \{x_1, x_2\}$. Note that $\{a, b_x, x_1, x_2\}$ is a cocircuit of M and X is a near-fan of M . Applying Corollary 3.4.7, we have that $\text{co}(M \setminus x_1)$ and $\text{co}(M \setminus x_2)$ are 3-connected so that (iii) holds.

Case (ii): X contains exactly one $b_c \in B$ such that $\text{si}(M/b_c)$ is 3-connected.

Now assume as stated that there is exactly one $b_c \in B \cap X$ such that $\text{si}(M/b_c)$ is 3-connected. If, for some $b_x \in (X \cap B) - \{b_c\}$, there is a vertical 3-separation $(S, \{b_x\}, T)$ of M such that S or T is contained in $(X \cup \{b\}) - \{b_c\}$, then there is such a vertical 3-separation such that $S \subseteq (X \cup \{b\}) - \{b_c\}$ and $T \cup \{b_x\}$ is closed. We may now apply case (i) to the vertical 3-separation $(S, \{b_x\}, T)$ to conclude that (iii) holds in addition to $\text{si}(M/b_c)$ being 3-connected.

So we may assume that if $(P, \{b_x\}, Q)$ is a vertical 3-separation of M with $b_x \in (X \cap B) - \{b_c\}$, then neither P nor Q is contained in $(X \cup \{b\}) - \{b_c\}$. Furthermore, by Lemma 3.3.6, we may assume that $Q \cup \{b_x\}$ is closed. Supposing that $(X \cap B) - \{b_c\} \neq \emptyset$, let $b_x \in (X \cap B) - \{b_c\}$, and let $(P, \{b_x\}, Q)$ be a vertical 3-separation with $Q \cup \{b_x\}$ closed. Without loss of generality, we may assume that $b \in Q$. Observe that 4.1.1.1, 4.1.1.2, 4.1.1.3, and 4.1.1.4 all hold. We deduce that $b_c \in X \cap P$. Letting $L_1 = \{b\} \cup (X \cap Q)$ and $L_2 = \{b_x\} \cup (X \cap P)$ as per the argument following 4.1.1.4, we again can assume that $|\text{cl}(L_1)| = |\text{cl}(L_2)| = 3$. Letting $L_2 - \{b_c, b_x\} = \{x_1\}$ it follows by an application of Theorem 3.4.4 that $\text{co}(M \setminus x_1)$ is 3-connected. Thus (ii) is satisfied.

The final possibility to consider is when $(X \cap B) - \{b_c\} = \emptyset$. Here $r(Y \cup \{b\}) = r(M) - 1$ so that X is a cocircuit with $r(X) = 3$. If $|X| \geq 4$, then Corollary 3.4.5 gives us (ii). So assume $|X| = 3$. Then X is a triad, and as M contains no 4-element fans, $X \cup \{b\}$ must be a circuit. Let $X - \{b_c\} = \{a, c\}$. If $\text{co}(M \setminus a)$ or $\text{co}(M \setminus c)$ is 3-connected, then (ii) holds. Otherwise, we have (iv). We conclude that the theorem holds. \square

The next result is an extension of Theorem 4.1.1 restated in a form which will be useful in what follows.

Theorem 4.1.2. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M such that for some $b \in B$, $\text{si}(M/b)$ is not 3-connected. Let $(X, \{b\}, Y)$ be a vertical 3-separation of M with $Y \cup \{b\}$ closed. Let K denote the set of elements of M removable with respect to B . Then either:*

- (i) $|X \cap K| \geq 2$, or
- (ii) $|X \cap K| = 1$, X is a triad, there exists $r \in Y \cap K$ such that $X \cup \{r\}$ is a 4-point coline with $|(X \cup \{r\}) \cap B| = 2$, and $(Y \cup \{b\}) - \{r\}$ is closed.

Proof. Assume that (i) does not hold. Applying Theorem 4.1.1, it must be that 4.1.1 (iv) holds. So X is a triad, $X \cup \{b\}$ is a circuit, and $X \cap K = X \cap B$. Now apply Theorem 3.4.8. As $|X \cap K| = 1$, 3.4.8 (i) doesn't hold. If 3.4.8 (ii) holds, then $X \cup \{a'\}$ and $X \cup \{c'\}$ are 4-element fans of M ; which cannot be. So 3.4.8 (iii) must hold. Let $X \cup \{r\}$ be the given 4-point coline of M . It follows by Lemmas 3.2.3 and 2.5.6 that $|(X \cup \{r\}) \cap B| = 2$. Finally, as $z \in \text{cl}^*((X \cup \{r\}) - \{z\})$ for all $z \in X \cup \{r\}$, Lemma 2.5.4 implies that $(Y \cup \{b\}) - \{r\}$ is closed. \square

Corollary 4.1.3. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M such that for some $c \in E(M) - B$, $\text{co}(M \setminus c)$ is not 3-connected. Let $(S, \{c\}, T)$ be a cyclic 3-separation of M with $T \cup \{c\}$ coclosed. Let K denote the set of elements of M which are removable with respect to B . Then either:*

- (i) $|S \cap K| \geq 2$, or
- (ii) $|S \cap K| = 1$, S is a triangle, there exists $r \in T \cap K$ such that $S \cup \{r\}$ is a 4-point line with $|(T \cup \{r\}) \cap (E(M) - B)| = 2$, and $(T \cup \{c\}) - \{r\}$ is coclosed.

Proof. This follows by dualising Theorem 4.1.2 and using Lemmas 3.3.3, 3.5.3, and 3.5.5. \square

4.2 The Minimal Number of Removable Elements

We now extend the results from the previous section to place a lower bound on the number of removable elements in a suitable matroid as a whole. We conclude this chapter with a proof of Theorem 4.0.1.

Theorem 4.2.1. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M such that for some $b \in B$, $\text{si}(M/b)$ is not 3-connected. Let $(X, \{b\}, Y)$ be a vertical 3-separation of M with $Y \cup \{b\}$ closed. Let K denote the set of elements which are removable with respect to B . Then either:*

- (i) $|X \cap K| \geq 2$ and $|Y \cap K| \geq 2$, or
- (ii) $X \cap K = \{b_x\}$, $|Y \cap K| \geq 3$, and there exists $r \in Y \cap K$ such that $X \cup \{r\}$ is a 4-point coline, or
- (iii) $|X \cap K| \geq 3$, $Y \cap K = \{b_y\}$, and there exists $r \in X \cap K$ such that $(Y - \text{cl}(X)) \cup \{r\}$ is a 4-point coline.

Proof. Note that the vertical 3-separation $(X, \{b\}, Y)$ satisfies the hypotheses of Theorem 4.1.2. Applying the theorem, either $|X \cap K| \geq 2$ or $|X \cap K| = 1$. Assume $|X \cap K| = 1$. Then X is a triad, there exists $r \in Y \cap K$ such that $X \cup \{r\}$ is a 4-point coline with $|(X \cup \{r\}) \cap B| = 2$, and $(Y \cup \{b\}) - \{r\}$ is closed. If $b \notin \text{cl}(Y - \{r\})$, then by Lemma 2.5.4, $b \in \text{cl}^*(X \cup \{r\})$, so that $X \cup \{b, r\}$ is a 5-point coline; but then $\text{si}(M/b)$ is 3-connected by the dual of Lemma 3.2.2: a contradiction. So $b \in \text{cl}(Y - \{r\})$.

Suppose $r(Y - \{r\}) = 2$. If $|Y - \{r\}| = 2$, $|Y| = 3$, and as $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is a vertical 3-separation by Lemma 3.3.6, it must be the case that $Y - \text{cl}(X) = Y$, so that $X \cup \{b\}$ is closed and Y is a triad. But now $(Y - \{r\}) \cup \{b\}$ is a triangle, and $Y \cup \{b\}$ is a contradictory 4-element fan. So if $r(Y - \{r\}) = 2$, we must have $|Y - \{r\}| \geq 3$. As $b \in \text{cl}(Y - \{r\})$, $|(Y - \{r\}) \cap (E(M) - B)| \geq 2$. Thus $|(Y - \{r\}) \cap K| \geq 2$ by Lemma 3.2.2. So $|Y \cap K| \geq 3$ and (ii) holds.

Now suppose that $r(Y - \{r\}) \geq 3$. By Lemma 3.2.8, $(X \cup \{r\}, \{b\}, Y - \{r\})$ is a vertical 3-separation of M , therefore by Lemma 3.3.6, $(\text{cl}(X \cup \{r\}) -$

$\{b\}, \{b\}, Y - \text{cl}(X \cup \{r\})$ is also. Note that $\text{cl}(\text{cl}(X \cup \{r\}) - \{b\}) \cup \{b\} = \text{cl}(X \cup \{r\})$ is closed. Now applying Theorem 4.1.2, if 4.1.2 (i) holds, then $|(Y - \text{cl}(X \cup \{r\})) \cap K| \geq 2$, so that $|Y \cap K| \geq 3$: (ii) is satisfied and the theorem holds. So assume that 4.1.2 (ii) holds. We have $|(Y - \text{cl}(X \cup \{r\})) \cap K| = 1$, $Y - \text{cl}(X \cup \{r\})$ is a triad, and there exists some $r' \in (\text{cl}(X \cup \{r\}) - \{b\}) \cap K$ such that $(Y - \text{cl}(X \cup \{r\})) \cup \{r'\}$ is a 4-point coline containing exactly two basis elements. Assume that $r' \in X \cup \{r\}$. Then, as $r' \in K$, we have two 4-point colines $X \cup \{r\}$ and $(Y - \text{cl}(X \cup \{r\})) \cup \{r'\}$ such that $|(X \cup \{r\}) \cap B| = |((Y - \text{cl}(X \cup \{r\})) \cup \{r'\}) \cap B| = 2$ and $(X \cup \{r\}) \cap ((Y - \text{cl}(X \cup \{r\})) \cup \{r'\}) = \{r'\} \in B$. This is a contradiction to Lemma 3.2.5. Thus $r' \notin X \cup \{r\}$ and $r' \in Y$, so that $|Y \cap K| \geq 3$ and (ii) holds.

Now assume that $|X \cap K| \geq 2$. If $|Y \cap K| \geq 2$, (i) holds. So assume that $|Y \cap K| \in \{0, 1\}$. $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is a vertical 3-separation by Lemma 3.3.6 with $(\text{cl}(X) - \{b\}) \cup \{b\} = \text{cl}(X)$ closed. Apply Theorem 4.1.2. If 4.1.2 (i) holds, $|(Y - \text{cl}(X)) \cap K| \geq 2$. But we're working under the assumption that $|Y \cap K| \leq 1$. So 4.1.2 (ii) holds. Then $|(Y - \text{cl}(X)) \cap K| = 1$, and there exists some $\rho \in (\text{cl}(X) - \{b\}) \cap K$ such that $(Y - \text{cl}(X)) \cup \{\rho\}$ is a 4-point coline containing exactly two basis elements. Now $|Y \cap K| = 1$, so $\rho \in X$. If $|X \cap K| \geq 3$, (iii) holds. So we may assume that $|X \cap K| = 2$. By Lemma 3.2.8, both $X - \{\rho\}$ and $(X - \{\rho\}) \cup \{b\}$ are exactly 3-separating. So $b \in \text{cl}^{(*)}(X - \{\rho\})$ by Lemma 3.2.7. Now $b \in \text{cl}(Y) \subset \text{cl}(Y \cup \{\rho\})$, so $b \notin \text{cl}^{(*)}(X - \{\rho\})$ by Lemma 2.5.4. Thus $b \in \text{cl}(X - \{\rho\})$.

Suppose $r(X - \{\rho\}) = 2$. If $|X - \{\rho\}| \geq 3$, then $(X - \{\rho\}) \cup \{b\}$ is a 4-point line. This implies that $|((X - \{\rho\}) \cup \{b\}) \cap (E(M) - B)| \geq 2$ by Lemma 2.3.3, so $|((X - \{\rho\}) \cup \{b\}) \cap K| \geq 2$, hence $|X - \{\rho\} \cap K| \geq 2$ and $|X \cap K| \geq 3$ giving (iii). Otherwise, $|X - \{\rho\}| = 2$. In this case, $(X - \{\rho\}) \cup \{b\}$ is a triangle, whereas X is a triad; giving a 4-element fan $X \cup \{b\}$ in M ; a contradiction.

Finally, suppose $r(X - \{\rho\}) \geq 3$. Then $(X - \{\rho\}, \{b\}, Y \cup \{\rho\})$ is a vertical 3-separation. By Lemma 3.3.6, $(X - \text{cl}(Y \cup \{\rho\}), \{b\}, \text{cl}(Y \cup \{\rho\}) - \{b\})$ is a vertical 3-separation also. Here $\text{cl}((\text{cl}(Y \cup \{\rho\}) - \{b\}) \cup \{b\}) = \text{cl}(Y \cup \{\rho\})$ is closed. Applying Theorem 4.1.2 and noting that $|X \cap K| = 2$ and $|(X - \{\rho\}) \cap K| = 1$, it follows that 4.1.2 (ii) holds. That is: $X - \text{cl}(Y \cup \{\rho\})$ is a triad, and there exists some $\rho' \in (\text{cl}(Y \cup \{\rho\}) - \{b\}) \cap K$ such that $(X - \text{cl}(Y \cup \{\rho\})) \cup \{\rho'\}$ is a 4-point coline containing exactly two basis elements. Note that $\rho' \in K$, $|X \cap K| = 2$, and $|Y \cap K| = |(Y - \text{cl}(X)) \cap$

$|K| = 1$. Therefore either $\rho' = \rho$ or $\rho' \in Y - \text{cl}(X)$. But in either case $(Y - \text{cl}(X)) \cup \{\rho\}$ and $(X - \text{cl}(Y \cup \{\rho\})) \cup \{\rho'\}$ are two 4-point colines such that $|((Y - \text{cl}(X)) \cup \{\rho\}) \cap B| = |((X - \text{cl}(Y \cup \{\rho\})) \cup \{\rho'\}) \cap B| = 2$, and $((Y - \text{cl}(X)) \cup \{\rho\}) \cap ((X - \text{cl}(Y \cup \{\rho\})) \cup \{\rho'\}) \subset B$. This contradicts Lemma 3.2.5. We conclude that the theorem holds. \square

We obtain the following as a corollary.

Corollary 4.2.2. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M such that for some $c \in (E(M) - B)$, $\text{co}(M \setminus c)$ is not 3-connected. Let $(S, \{c\}, T)$ be a cyclic 3-separation of M with $T \cup \{c\}$ coclosed. Let K denote the set of elements which are removable with respect to B . Then either:*

- (i) $|S \cap K| \geq 2$ and $|T \cap K| \geq 2$, or
- (ii) $S \cap K = \{c_s\}$, $|T \cap K| \geq 3$, and there exists $r \in T \cap K$ such that $S \cup \{r\}$ is a 4-point line, or
- (iii) $|S \cap K| \geq 3$, $T \cap K = \{c_t\}$, and there exists $r \in S \cap K$ such that $(T - \text{cl}^*(S)) \cup \{r\}$ is a 4-point line.

Proof. This follows by dualising Theorem 4.2.1 and using Lemmas 3.3.3, 3.5.3, and 3.5.5. \square

Suppose M is a 3-connected matroid with no 4-element fans, B a basis of M , and let K denote the set of elements of M which are removable with respect to B . Theorem 4.2.1 states that if there exists some $b \in B - K$, then $|K| \geq 4$. Also, Corollary 4.2.2 states that if there exists some $c \in (E(M) - B) - K$, then $|K| \geq 4$. Putting these results together we obtain the following:

Proof of Theorem 4.0.1. Suppose $|K| \leq 3$. Then by Theorem 4.2.1 and Corollary 4.2.2, $B - K = (E(M) - B) - K = \emptyset$. But then $B \cup (E(M) - B) = E(M) = K$ so that $|E(M)| \leq 3$. \square

Chapter 5

Matroids with Minimal Removability

In light of the results in preceding chapter, we have that for any matroid M on a ground set of size at least four, and for any basis B of M , provided that M does not contain any four element fans, we can always find *at least* four elements which are removable with respect to B . We now turn our attention to the cases for which there exist *exactly* four elements which are removable with respect to B . We begin with a definition.

Definition 5.0.1. A matroid M has *path-width 3* if there is an ordering (e_1, \dots, e_n) of $E(M)$ such that, for all $i \in \{1, \dots, n\}$, $\lambda(\{e_1, \dots, e_i\}) \leq 2$

The class of matroids with path-width 3 has been studied in detail, and as a result, the structure of matroids which belong to this class is well understood. See Hall, Oxley, and Semple [6] for a detailed examination of this class of matroids. The primary result of this chapter is the following:

Theorem 5.0.2. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M . Let K denote the set of elements which are removable with respect to B . If $|K| = 4$, then M has path-width 3.*

5.1 Preliminary Constructions

Proposition 5.1.1. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M such that for some $b \in B$, $\text{si}(M/b)$ is not 3-connected.*

Let K denote the set of elements which are removable with respect to B . Suppose $|K| = 4$. Then, provided M is not the union of a 4-point line and a 4-point coline, there exists a vertical 3-separation $(X, \{b\}, Y)$ of M such that $|X \cap K| = 2$, $|Y \cap K| = 2$, and $Y \cup \{b\}$ is closed.

Proof. Let $(X', \{b\}, Y')$ be a vertical 3-separation of M with $Y' \cup \{b\}$ closed. If $|X' \cap K| = |Y' \cap K| = 2$, the result is immediate. So assume that this isn't the case. Now either 4.2.1 (ii) or 4.2.1 (iii) holds.

Suppose (ii) holds. Then $X' \cap K = \{b_x\}$, $|Y' \cap K| \geq 3$, and there exists some $r \in Y' \cap K$ such that $X' \cup \{r\}$ is a 4-point coline. By Lemma 3.4.3, either $(X' \cup \{r\}, \{b\}, Y' - \{r\})$ is a vertical 3-separation of M , or $r((Y' - \{r\}) \cup \{b\}) = 2$. Suppose the latter holds. If $|Y' - \{r\}| \geq 4$, then $|((Y' - \{r\}) \cup \{b\}) \cap K| \geq 3$ by Lemmas 2.3.3 and 3.2.2. But then $|Y' \cap K| \geq 4$; which is absurd. If $|Y' - \{r\}| = 3$, then $(Y' - \{r\}) \cup \{b\}$ is a 4-point line. But $E - ((Y' - \{r\}) \cup \{b\}) = X \cup \{r\}$ is a 4-point coline, so that M is the union of a 4-point line and a 4-point coline, which is not the case. So it must be that $|Y' - \{r\}| = 2$. Here $|Y'| = 3$. As $(\text{cl}(X') - \{b\}, \{b\}, Y' - \text{cl}(X'))$ is a vertical 3-separation of M by Lemma 3.3.6, it must be that $\text{cl}(X' - \{b\}) \cup \{b\} = \text{cl}(X') = X'$ is closed. Therefore, Y' is a triad that meets the triangle $(Y' - \{r\}) \cup \{b\}$; giving a 4-element fan $Y' \cup \{b\}$ in M ; a contradiction. So $(X' \cup \{r\}, \{b\}, Y' - \{r\})$ must be a vertical 3-separation of M . As $X' \cup \{r\}$ is a 4-point coline, $x \in \text{cl}^*((X' \cup \{r\}) - \{x\})$ for all $x \in X' \cup \{r\}$. Hence $\text{cl}((Y' - \{r\}) \cup \{b\}) \cap (X' \cup \{r\}) = \emptyset$ and $(Y' - \{r\}) \cup \{b\}$ is closed. Letting $Y = Y' - \{r\}$ and $X = X' \cup \{r\}$, the result follows.

Now suppose 4.2.1 (iii) holds. Then $|X' \cap K| \geq 3$, $Y' \cap K = \{b_y\}$, and there exists some $r \in X' \cap K$ such that $(Y' - \text{cl}(X')) \cup \{r\}$ is a 4-point coline. Now $(\text{cl}(X') - \{b\}, \{b\}, Y' - \text{cl}(X'))$ is a vertical 3-separation of M by Lemma 3.3.6. By Lemma 3.4.3, either $(\text{cl}(X') - \{b, r\}, \{b\}, (Y' - \text{cl}(X')) \cup \{r\})$ is a vertical 3-separation, or $r((\text{cl}(X') - \{b, r\}) \cup \{b\}) = 2$. Suppose $r((\text{cl}(X') - \{b, r\}) \cup \{b\}) = 2$. If $|\text{cl}(X') - \{b, r\}| = 2$, then $|\text{cl}(X') - \{b\}| = 3$ and so $\text{cl}(X') - \{b\} = X'$. As $Y' \cup \{b\}$ is closed, X' is a triad, whereas $(X' - \{r\}) \cup \{b\}$ is a triangle. This gives a contradictory 4-element fan $X' \cup \{b\}$ in M . If $|\text{cl}(X') - \{b, r\}| = 3$, then $\text{cl}(X') - \{r\}$ is a 4-point line, whereas $(Y' - \text{cl}(X')) \cup \{r\}$ is a 4-point coline. But M is not the union of a 4-point line and a 4-point coline. Finally, if $|\text{cl}(X') - \{b, r\}| \geq 4$, then $\text{cl}(X') - \{r\}$ is a rank 2 set containing at least five elements. Therefore $|(\text{cl}(X') - \{r\}) \cap K| \geq 3$ by Lemmas 2.3.3 and 3.2.2. As $b_y \in Y' - \text{cl}(X')$, $b_y \notin (\text{cl}(X') - \{r\}) \cap K$.

Thus $(\text{cl}(X') - \{r\}) \cap K \subset X'$, and $|X' \cap K| \geq 4$; a contradiction. It must be that $(\text{cl}(X') - \{b, r\}, \{b\}, (Y' - \text{cl}(X')) \cup \{r\})$ is a vertical 3-separation of M . Moreover, as $r \in \text{cl}^*(Y' - \text{cl}(X'))$, $r \notin \text{cl}((\text{cl}(X') - \{b, r\}) \cup \{b\})$. Also, $(Y' - \text{cl}(X')) \cap \text{cl}((\text{cl}(X') - \{b, r\}) \cup \{b\}) = \emptyset$. Therefore, $(\text{cl}(X') - \{b, r\}) \cup \{b\}$ is closed. Letting $X = (Y' - \text{cl}(X')) \cup \{r\}$ and $Y = \text{cl}(X') - \{b, r\}$, we conclude that the theorem holds. \square

Let M be a 3-connected matroid, let B be a basis of M , and let K denote the set of elements of $E(M)$ which are removable with respect to B . We shall call a vertical (cyclic) 3-separation $(X, \{e\}, Y)$ of M *balanced* whenever $|X \cap K| = |Y \cap K| = 2$.

The following is a simple restatement of Proposition 5.1.1 using this terminology.

Corollary 5.1.2. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M . Let K denote the set of elements which are removable with respect to B , and suppose $|K| = 4$. If M is not the union of a 4-point line and a 4-point coline, then for each $b_i \in B - K$, there exists a balanced vertical 3-separation $(X_i, \{b_i\}, Y_i)$ such that $Y_i \cup \{b_i\}$ is closed.*

Corollary 5.1.3. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M . Let K denote the set of elements which are removable with respect to B , and suppose $|K| = 4$. If M is not the union of a 4-point line and a 4-point coline, then for each $c_i \in (E(M) - B) - K$, there exists a balanced cyclic 3-separation $(S_i, \{c_i\}, T_i)$ such that $T_i \cup \{c_i\}$ is coclosed.*

Proof. This follows from the dual of Corollary 5.1.2 and the use of Lemmas 3.5.3, and 3.5.5. \square

Corollary 5.1.2 and Corollary 5.1.3 above are used freely throughout the remainder of this thesis.

Corollary 5.1.4. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M . Let K denote the set of elements which are removable with respect to B , and suppose $|K| = 4$. If $|K \cap (E(M) - B)| \geq 3$, then for every $b \in B - K$, every vertical 3-separation $(X, \{b\}, Y)$ of M is balanced.*

Proof. Suppose $|K \cap (E(M) - B)| \geq 3$ and $b \in B - K$. Let $(X, \{b\}, Y)$ be a vertical 3-separation of M that is not balanced. Then we can assume that $|X \cap K| = 1$. Now $(X - \text{cl}(Y), \{b\}, \text{cl}(Y) - \{b\})$ is also a vertical 3-separation of M by Lemma 3.3.6, and, by Theorem 4.1.2, $|(X - \text{cl}(Y)) \cap K| = 1$, and there exists some $r \in \text{cl}(Y) \cap K$ such that $(X - \text{cl}(Y)) \cup \{r\}$ is a 4-point coline. But then $|((X - \text{cl}(Y)) \cup \{r\}) \cap B \cap K| \geq 2$ by Lemmas 2.5.6 and 3.2.3. This contradicts the fact that $|K \cap B| \in \{0, 1\}$. Therefore $(X, \{b\}, Y)$ is balanced. \square

Lemma 5.1.5. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M . Let K denote the set of elements which are removable with respect to B , and suppose $|K| = 4$. Then:*

- (i) *if L is a line of M and there exists some $b \in B - K$, then $|L| \leq 4$, and*
- (ii) *if Γ is a coline of M and there exists some $c \in (E(M) - B) - K$, then $|\Gamma| \leq 4$.*

Proof. Suppose $b \in B - K$ and let L be a line of M with $|L| \geq 5$. Then $|L \cap (E(M) - B) \cap K| \geq 3$ by Lemmas 3.2.2 and 2.3.3. Let $(X, \{b\}, Y)$ be a vertical 3-separation of M . We can assume without loss of generality that $|L \cap Y| \geq 2$. By Lemma 3.3.6, $(X - \text{cl}(Y), \{b\}, \text{cl}(Y) - \{b\})$ is a vertical 3-separation of M , and we have $L \cap (E(M) - B) \cap K \subset \text{cl}(Y) - \{b\}$. So $(X - \text{cl}(Y), \{b\}, \text{cl}(Y) - \{b\})$ is not balanced. But this contradicts Lemma 5.1.4. Therefore $|L| \leq 4$ and (i) holds. By dualising this result and using Lemma 3.5.3, we obtain (ii). \square

Proposition 5.1.6. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M . Let K denote the set of elements which are removable with respect to B , and suppose $|K| = 4$. Let Γ_1, Γ_2 be distinct 4-point colines of M . If there exists some $c \in (E(M) - B) - K$, then $|\Gamma_1 \cap \Gamma_2 \cap B| \neq 1$.*

Proof. Suppose that $c \in (E(M) - B) - K$ and $|\Gamma_1 \cap \Gamma_2 \cap B| = 1$. By Lemma 2.5.6, $|\Gamma_1 \cap B| \geq 2$ and $|\Gamma_2 \cap B| \geq 2$. If $|\Gamma_1 \cap \Gamma_2| \geq 2$, then $r^*(\Gamma_1 \cap \Gamma_2) \geq 2$ as M is 3-connected, and so, by Lemma 2.5.7, $\Gamma_1 \cup \Gamma_2$ is a coline containing at least five elements. But this contradicts Lemma 5.1.5. So we may assume that $\Gamma_1 \cap \Gamma_2 = \Gamma_1 \cap \Gamma_2 \cap B$.

By Lemma 3.2.5, it cannot be the case that $|\Gamma_1 \cap B| = |\Gamma_2 \cap B| = 2$. So we can assume without loss of generality that $|\Gamma_1 \cap B| \geq 3$. Now $|(\Gamma_1 \cup \Gamma_2) \cap B| =$

$|(\Gamma_1 \cap B) \cup (\Gamma_2 \cap B)| = |\Gamma_1 \cap B| + |\Gamma_2 \cap B| - |\Gamma_1 \cap \Gamma_2 \cap B| \geq 4$. As $|K| = 4$, and every element of $(\Gamma_1 \cup \Gamma_2) \cap B$ is removable with respect to B , it must be that $|(\Gamma_1 \cup \Gamma_2) \cap B| = 4$ and $K = (\Gamma_1 \cup \Gamma_2) \cap B$. In particular, note that $K \subseteq B$.

Now consider the matroid M^* with basis $E(M) - B$. As M is 3-connected, M^* is also 3-connected. Also, M^* contains no 4-element fans by Lemma 3.5.5. By Lemma 3.5.3, the set of elements removable with respect to $E(M) - B$ is equal to K . Γ_1 and Γ_2 are 4-point lines of M^* which meet at exactly one point $\alpha \in K$. Note that $r^*(\Gamma_1 \cup \Gamma_2) = 3$. Consider the element $c \in E(M) - B$. As $c \notin K$, $\text{si}(M^*/c)$ is not 3-connected. By Lemmas 3.3.12 and 3.3.6, there exists a vertical 3-separation $(P, \{c\}, Q)$ of M^* with $Q \cup \{c\}$ closed. Recall that $|\Gamma_1 \cap K| = 3$ and $K \subseteq B$. Note that B is now a cobasis of the matroid M^* so that Corollary 5.1.4 may be used freely when studying M^* . If $|\Gamma_1 \cap Q| \geq 2$, then $\Gamma_1 \subseteq Q \cup \{c\}$ as $Q \cup \{c\}$ is closed. But then $|(Q \cup \{c\}) \cap K| \geq 3$ which contradicts Lemma 5.1.4. Therefore $|\Gamma_1 \cap Q| \leq 1$. So $|\Gamma_1 \cap P| \geq 2$. By Lemma 3.3.6, $(\text{cl}(P) - \{c\}, \{c\}, Q - \text{cl}(P))$ is a vertical 3-separation of M^* . But now $\Gamma_1 \subset \text{cl}(P) - \{c\}$ so that $(\text{cl}(P) - \{c\}, \{c\}, Q - \text{cl}(P))$ is not balanced; contradicting Corollary 5.1.4. We conclude that the proposition holds. \square

Lemma 5.1.7. *Let M be a 3-connected matroid with no 4-element fans and let B be a basis of M . Let K denote the set of elements which are removable with respect to B , and suppose $|K| = 4$. If $b \in B - K$ and $(X, \{b\}, Y)$ is a balanced vertical 3-separation of M with $Y \cup \{b\}$ closed, then for any $w \in X \cap \text{cl}^*(Y)$, either:*

- (i) $(X - \{w\}, \{b\}, Y \cup \{w\})$ is a vertical 3-separation of M , or
- (ii) $(X - \{w\}) \cup \{b\}$ is a 4-point line.

Proof. Let w be an element of $X \cap \text{cl}^*(Y)$. Suppose $r((X - \{w\}) \cup \{b\}) = 2$. If $|(X - \{w\}) \cup \{b\}| \geq 5$, then $|(X - \{w\}) \cap K| \geq 3$ by Lemmas 2.3.3 and 3.2.2; but this contradicts the fact that $(X, \{b\}, Y)$ is balanced. If $|(X - \{w\}) \cup \{b\}| = 3$, then $|X| = 3$, so that X is a triad and $X \cup \{b\}$ is a contradictory 4-element fan. Therefore $r((X - \{w\}) \cup \{b\}) = 2 \implies |(X - \{w\}) \cup \{b\}| = 4$; giving (ii). Otherwise, $r((X - \{w\}) \cup \{b\}) \geq 3$ and by Lemma 3.4.3, $(X - \{w\}, \{b\}, Y \cup \{w\})$ is a vertical 3-separation of M . \square

Let M be a 3-connected matroid, B a basis of M , and let K denote the set of elements which are removable with respect to B . We shall call a vertical (cyclic) 3-separation $(X, \{b\}, Y)$ of M *colinearly balanced* if it is balanced, and, for any 4-point coline $\Gamma \subset E(M)$, $\Gamma \subset X$ or $\Gamma \subset Y$. Similarly, we shall call a vertical (cyclic) 3-separation $(X, \{b\}, Y)$ *linearly balanced* if it is balanced, and, for any 4-point line $L \subset E(M)$, $L \subset X$ or $L \subset Y$.

Let M be a matroid, and B be a basis of M . We shall call M *colinearly entwined* if there exists 4-point colines $\Gamma_1, \Gamma_2 \subset E(M)$ such that $|\Gamma_1 \cap \Gamma_2| = 1$ where $\Gamma_1 \cap \Gamma_2 \subseteq E(M) - B$.

Proposition 5.1.8. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M . Suppose that M is colinearly entwined with respect to the 4-point colines $\Gamma_1, \Gamma_2 \subset E(M)$. Let K denote the set of elements which are removable with respect to B , and suppose $|K| = 4$. Then for all $b \in B - K$, there exists a balanced vertical 3-separation $(X', \{b\}, Y')$ of M with $\Gamma_1 \subseteq X'$ and $\Gamma_2 - \Gamma_1 \subseteq Y'$.*

Proof. Let $\Gamma_1 \cap \Gamma_2 = \{\alpha\}$. Then $\alpha \in E(M) - B$. It follows from Lemma 3.2.3 and Lemma 2.5.6 that $K \subset \Gamma_1 \cup \Gamma_2$, $\alpha \notin K$, and M does not contain any 4-point colines apart from Γ_1 and Γ_2 . Suppose $b \in B - K$, and let $(X, \{b\}, Y)$ be a balanced vertical 3-separation of M with $Y \cup \{b\}$ closed. We construct a balanced vertical 3-separation that satisfies the theorem through a case analysis of the possible sizes of the set $\Gamma_1 \cap Y$.

Case (i): $|\Gamma_1 \cap Y| = 4$

Suppose $\Gamma_1 \cap X = \emptyset$. Then $|\Gamma_2 \cap X \cap K| = 2$. Hence $\Gamma_2 \subseteq \text{cl}^*(X)$. Note that $|(\Gamma_1 \cap Y) - \Gamma_2| = 3$ so that $r(Y - \Gamma_2) \geq 3$ by Lemma 3.2.4. Applying Lemma 3.4.3, it follows that $(X \cup \Gamma_2, \{b\}, Y - \Gamma_2)$ is a suitable balanced vertical 3-separation of M .

Case (ii): $|\Gamma_1 \cap Y| = 3$

Suppose $\Gamma_1 \cap X = \{z\}$. Then $(X - \{z\}, \{b\}, Y \cup \{z\})$ is a vertical 3-separation of M by Lemma 5.1.7. If $(X - \{z\}, \{b\}, Y \cup \{z\})$ is not balanced, then by Theorem 4.1.2, $X - \{z\}$ contains a triad Λ and there exists some $r \in (Y \cup \{z\}) \cap B$ such that $\Lambda \cup \{r\}$ is a 4-point coline. But then it must be that $\Lambda \cup \{r\} = \Gamma_2$ and $r = \alpha$ which is absurd as $\alpha \notin B$. So $(X -$

$\{z\}, \{b\}, Y \cup \{z\}$) must be balanced. Then $|(X - \{z\}) \cap \Gamma_2 \cap K| = 2$. So $\Gamma_2 \subseteq \text{cl}^*(X - \{z\})$. Now $\Gamma_1 - \{\alpha\} \subseteq (Y \cup \{z\}) - \Gamma_2$, hence $r((Y \cup \{z\}) - \Gamma_2) \geq 3$ by Lemma 3.2.4. Therefore, applying Lemma 3.4.3, it must be that $((X - \{z\}) \cup (\Gamma_2 - \{\alpha\}), \{b\}, (Y \cup \{z\}) - (\Gamma_2 - \{\alpha\}))$ is a vertical 3-separation of M . Letting $X' = (Y \cup \{z\}) - (\Gamma_2 - \{\alpha\})$ and $Y' = (X - \{z\}) \cup (\Gamma_2 - \{\alpha\})$, the proposition follows.

Case (iii): $|\Gamma_1 \cap Y| = 2$

Suppose $\Gamma_1 \cap X = \{z_1, z_2\}$. By Lemma 5.1.7, $(X - \{z_1\}, \{b\}, Y \cup \{z_1\})$ is a vertical 3-separation of M . Applying Lemma 3.3.6, $(X - \text{cl}(Y \cup \{z_1\}), \{b\}, \text{cl}(Y \cup \{z_1\}) - \{b\})$ is also a vertical 3 separation of M . As $z_2 \in \text{cl}^*(Y \cup \{z_1\})$, Lemma 3.3.7 implies that $z_2 \notin \text{cl}(Y \cup \{z_1\})$.

If $(X - \text{cl}(Y \cup \{z_1\}), \{b\}, \text{cl}(Y \cup \{z_1\}) - \{b\})$ is not balanced, then by Theorem 4.1.2, $X - \text{cl}(Y \cup \{z_1\})$ is a triad and there exists some $r \in \text{cl}(Y \cup \{z_1\}) - \{b\}$ such that $(X - \text{cl}(Y \cup \{z_1\})) \cup \{r\}$ is a 4-point coline. Now $z_2 \in (X - \text{cl}(Y \cup \{z_1\})) \cup \{r\}$ and $z_2 \in \Gamma_1$, so $z_2 = \alpha$ and $r \notin \Gamma_1$. Therefore $r(\text{cl}(Y \cup \{z_1\}) - \{b, r\}) \geq 3$ by Lemma 3.2.4, and so, by an application of Lemma 3.4.3, it must be that $((X - \text{cl}(Y \cup \{z_1\})) \cup \{r\}, \{b\}, \text{cl}(Y \cup \{z_1\}) - \{b, r\})$ is a vertical 3-separation of M . Letting $X' = (X - \text{cl}(Y \cup \{z_1\})) \cup \{r\}$ and $Y' = \text{cl}(Y \cup \{z_1\}) - \{b, r\}$, we obtain a suitable balanced vertical 3-separation of M .

So assume that $(X - \text{cl}(Y \cup \{z_1\}), \{b\}, \text{cl}(Y \cup \{z_1\}) - \{b\})$ is balanced. Noting that $(\text{cl}(Y \cup \{z_1\}) - \{b\}) \cup \{b\} = \text{cl}(Y \cup \{z_1\})$ is closed, and applying Lemma 5.1.7, we have that $(X - (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\}), \{b\}, (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\}) - \{b\})$ is a vertical 3-separation of M . If this separation is not balanced, then by Theorem 4.1.2, $X - (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\})$ contains a triad Λ and there exists some $r' \in (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\}) - \{b\}$ where $r' \in B$ such that $\Lambda \cup \{r'\}$ is a 4-point coline. But then $\Lambda \cup \{r'\} = \Gamma_2$ and $r' = \alpha$. This contradicts the fact that $\alpha \notin B$. So $(X - (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\}), \{b\}, (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\}) - \{b\})$ must be balanced. Hence $|\Gamma_2 \cap (X - (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\}))| \geq 2$. Therefore $\Gamma_2 \in \text{cl}^*(X - (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\}))$. Noting that $r((\text{cl}(Y \cup \{z_1\}) \cup \{z_2\}) - (\{b\} \cup \Gamma_2)) \geq 3$ by Lemma 3.2.4, we have that, by Lemma 3.4.3, $((X - (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\})) \cup \Gamma_2, \{b\}, ((Y \cup \{z_1\}) \cup \{z_2\}) - (\{b\} \cup \Gamma_2))$ is a vertical 3-separation of M . Letting $X' = (X - (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\})) \cup \Gamma_2$ and $Y' = ((Y \cup \{z_1\}) \cup \{z_2\}) - (\{b\} \cup \Gamma_2)$, the result follows.

Case (iv): $|\Gamma_1 \cap Y| = 1$

Suppose $\Gamma_1 \cap Y = \{z\}$. Then $z \in \text{cl}^*(X)$ and so $z \notin \text{cl}(X)$ by Lemma 3.3.7. Applying Lemma 3.3.6, $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is a vertical 3-separation of M with $z \in Y - \text{cl}(X)$. If it isn't balanced, then by Theorem 4.1.2, $Y - \text{cl}(X)$ is a triad and there exists some $r \in \text{cl}(X) - \{b\}$ such that $(Y - \text{cl}(X)) \cup \{r\}$ is a 4-point coline. Then $(Y - \text{cl}(X)) \cup \{r\} = \Gamma_2$, $z = \alpha$, and it must be that $r \notin \Gamma_1$ because $|\Gamma_1 \cap \Gamma_2| = 1$. So $r(\text{cl}(X) - \{b, r\}) \geq 3$ by Lemma 3.2.4. Now $r \in \text{cl}^*(Y - \text{cl}(X))$, so that, by Lemma 3.4.3, $(\text{cl}(X) - \{b, r\}, \{b\}, (Y - \text{cl}(X)) \cup \{r\})$ is a vertical 3-separation of M . Letting $X' = (Y - \text{cl}(X)) \cup \{r\}$ and $Y' = \text{cl}(X) - \{b, r\}$, we have $(X', \{b\}, Y')$ as a suitable balanced vertical 3-separation of M .

So assume that $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is balanced. Then by Lemma 5.1.7, $((\text{cl}(X) \cup \{z\}) - \{b\}, \{b\}, Y - (\text{cl}(X) \cup \{z\}))$ is a vertical 3-separation of M . If this 3-separation is not balanced, then by Theorem 4.1.2, $Y - (\text{cl}(X) \cup \{z\})$ contains a triad Λ and there exists some $r' \in (\text{cl}(X) \cup \{z\}) - \{b\}$ where $r' \in B$ such that $\Lambda \cup \{r'\}$ is a 4-point coline. But then $\Lambda \cup \{r'\} = \Gamma_2$ and $r' = \alpha$. This contradicts the fact that $\alpha \notin B$. Hence $((\text{cl}(X) \cup \{z\}) - \{b\}, \{b\}, Y - (\text{cl}(X) \cup \{z\}))$ must be balanced. Therefore $|(Y - (\text{cl}(X) \cup \{z\})) \cap \Gamma_2 \cap K| = 2$ and $\Gamma_2 \subseteq \text{cl}^*(Y - (\text{cl}(X) \cup \{z\}))$. Now $|\Gamma_1 \cap ((\text{cl}(X) \cup \{z\}) - (\{b\} \cup \{\Gamma_2\}))| = 3$, so that $r((\text{cl}(X) \cup \{z\}) - (\{b\} \cup \{\Gamma_2\})) \geq 3$. With Lemma 3.4.3, we then deduce that $((\text{cl}(X) \cup \{z\}) - (\{b\} \cup \{\Gamma_2\}), \{b\}, (Y - (\text{cl}(X) \cup \{z\})) \cup \Gamma_2)$ is a vertical 3-separation of M . Let $X' = (Y - (\text{cl}(X) \cup \{z\})) \cup \Gamma_2$ and $Y' = (\text{cl}(X) \cup \{z\}) - (\{b\} \cup \{\Gamma_2\})$, and it follows that $(X', \{b\}, Y')$ is a suitable balanced vertical 3-separation of M .

Case (v): $|\Gamma_1 \cap Y| = 0$

Suppose $\Gamma_1 \subseteq X$. Then $|\Gamma_2 \cap Y \cap K| = 2$, so that $\Gamma_2 \subseteq \text{cl}^*(Y)$. We have $r(X - \Gamma_2) \geq 3$ so an application of Lemma 3.4.3 gives a balanced vertical 3-separation $(X - \Gamma_2, \{b\}, Y \cup \Gamma_2)$ that satisfies the requirements of the Proposition. \square

Proposition 5.1.9. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M . Let K denote the set of elements which are removable with respect to B . Suppose $|K| = 4$ and that there exists some $c \in (E(M) - B) - K$. If M is not colinearly entwined and M is not the*

union of a 4-point line and a 4-point coline, then for each $b_i \in B - K$ such that b_i is not on 4-point line, there exists a colinearly balanced vertical 3-separation $(X_i, \{b_i\}, Y_i)$ of M .

Proof. Suppose M is not colinearly entwined and M is not the union of a 4-point line and 4-point coline. Note that $b \notin \Gamma$ for any $b \in B - K$ and any 4-point coline Γ of M by Lemma 3.2.3.

5.1.9.1. *If Γ_1, Γ_2 are two distinct 4-point colines of M then $\Gamma_1 \cap \Gamma_2 = \emptyset$.*

Proof. By Lemma 5.1.6 and the fact that M is not colinearly entwined, $|\Gamma_1 \cap \Gamma_2| \neq 1$. If $|\Gamma_1 \cap \Gamma_2| \geq 2$, then $r^*(\Gamma_1 \cap \Gamma_2) \geq 2$, and by Lemma 2.5.7, either $\Gamma_1 = \Gamma_2$; which is not the case, or $\Gamma_1 \cup \Gamma_2$ is a coline of cardinality greater than 4; which contradicts Lemma 5.1.5. So $\Gamma_1 \cap \Gamma_2 = \emptyset$. \square

Suppose $b \in B - K$. Let $(X, \{b\}, Y)$ be a balanced vertical 3-separation of M with $Y \cup \{b\}$ closed. If M does not contain any 4-point colines, then any balanced vertical 3-separation of M is immediately colinearly balanced, and the theorem follows. So assume that Γ_1 is a 4-point coline of M . We shall construct a suitable colinearly balanced vertical 3-separation. The construction is a case analysis dependant on the size of $\Gamma_1 \cap Y$.

Case (i): $|\Gamma_1 \cap Y| = 4$

Suppose $\Gamma_1 \subseteq Y$. If Γ_1 is the only 4-point coline of M , then $(X, \{b\}, Y)$ is colinearly balanced. So let Γ_2 be second 4-point coline of M . As $(X, \{b\}, Y)$ is balanced, it must be the case that $|\Gamma_2 \cap X| \geq 2$ and $\Gamma_2 \cap Y \cap K = \emptyset$. By Lemma 3.2.4 and 5.1.9.1, $r(Y - \Gamma_2) \geq 4$. Note that $\Gamma_2 \subseteq \text{cl}^*(X)$. Applying Lemma 3.4.3, and noting that M contains at most two distinct 4-point colines, it follows that $(X \cup \Gamma_2, \{b\}, Y - \Gamma_2)$ is a colinearly balanced vertical 3-separation of M .

Case (ii): $|\Gamma_1 \cap Y| = 3$

Suppose that $|\Gamma_1 \cap Y| = 3$ with $\Gamma_1 \cap X = \{z_1\}$. Then, as b does not lie on a 4-point line, $(X - \{z_1\}, \{b\}, Y \cup \{z_1\})$ is a vertical 3-separation of M by Lemma 5.1.7. If this vertical 3-separation is not balanced, then by Lemma 4.1.2, $X - \{z_1\}$ contains a triad Λ and there exists some $r \in$

$Y \cup \{z_1\}$ such that $\Lambda \cup \{r\}$ is a 4-point coline. By 5.1.9.1, $r \notin \Gamma_1$. As $r \in \text{cl}^*(X - \{z_1\})$, an application of Lemma 3.4.3 yields a colinearly balanced vertical 3-separation $((X \cup \{r\}) - \{z_1\}, \{b\}, (Y \cup \{z_1\}) - \{r\})$ of M .

So assume that $(X - \{z_1\}, \{b\}, Y \cup \{z_1\})$ is balanced. If Γ_1 is the only 4-point line of M , then $(X - \{z_1\}, \{b\}, Y \cup \{z_1\})$ is colinearly balanced. So assume that M contains another 4-point coline Γ_2 . Then $|(X - \{z_1\}) \cap \Gamma_2| \geq 2$ because $K - \Gamma_1 \subset X \cap \Gamma_2$. So $\Gamma_2 - (X - \{z_1\}) \subset \text{cl}^*(X - \{z_1\})$. By 5.1.9.1 and Lemma 3.2.4, $r((Y \cup \{z_1\}) - \Gamma_2) \geq 4$, hence, by an application of Lemma 3.4.3, we conclude that $((X - \{z_1\}) \cup \Gamma_2, \{b\}, (Y \cup \{z_1\}) - \Gamma_2)$ is a colinearly balanced vertical 3-separation of M .

Case (iii): $|\Gamma_1 \cap Y| = 2$

Suppose $\Gamma_1 \cap X = \{z_1, z_2\}$. Then $(X - \{z_1\}, \{b\}, Y \cup \{z_1\})$ is a vertical 3-separation of M by Lemma 5.1.7. Now $z_2 \in \text{cl}^*(Y \cup \{b, z_1\})$, so $z_2 \notin \text{cl}(Y \cup \{b, z_1\})$ by Lemma 3.3.7. Therefore, by Lemma 3.3.6, $(X - \text{cl}(Y \cup \{z_1\}), \{b\}, \text{cl}(Y \cup \{z_1\}) - \{b\})$ is a vertical 3 separation of M with $z_2 \in X - \text{cl}(Y \cup \{z_1\})$. If this vertical 3-separation is not balanced, then by Theorem 4.1.2, $X - \text{cl}(Y \cup \{z_1\})$ is a triad contained in a 4-point coline. This contradicts 5.1.9.1. So $(X - \text{cl}(Y \cup \{z_1\}), \{b\}, \text{cl}(Y \cup \{z_1\}) - \{b\})$ must be balanced. Applying Lemma 5.1.7 once more, $(X - (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\}), \{b\}, \text{cl}(Y \cup \{z_1\}) \cup \{z_2\})$ is a vertical 3-separation of M .

If this vertical 3-separation isn't balanced, then by Theorem 4.1.2, $X - (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\})$ contains a triad Λ and there exists some $r \in \text{cl}(Y \cup \{z_1\}) \cup \{z_2\}$ such that $\Lambda \cup \{r\}$ is a 4-point coline. By 5.1.9.1, $r \notin \Gamma_1$. Therefore, $((X - (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\})) \cup \{r\}, \{b\}, (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\}) - \{r\})$ is a colinearly balanced vertical 3-separation of M by Lemmas 3.4.3 and 3.2.4. So assume that $(X - (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\}), \{b\}, \text{cl}(Y \cup \{z_1\}) \cup \{z_2\})$ is balanced. If Γ_1 is the only 4-point coline of M , we then have our colinearly balanced vertical 3-separation. So assume that Γ_2 is some other 4-point coline of M . It must be that $|\Gamma_2 \cap K \cap (X - (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\}))| = 2$. Now by 5.1.9.1 together with Lemmas 3.4.3 and 3.2.4, $((X - (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\})) \cup \Gamma_2, \{b\}, (\text{cl}(Y \cup \{z_1\}) \cup \{z_2\}) - \Gamma_2)$ is a colinearly balanced vertical 3-separation of M .

Case (iv): $|\Gamma_1 \cap Y| = 1$

Suppose $\Gamma_1 \cap Y = \{z_1\}$. Then $z_1 \in \text{cl}^*(X)$, so that $z_1 \notin \text{cl}(X)$ by Lemma 3.3.7. Hence $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is a vertical 3-separation with $z_1 \in (Y - \text{cl}(X))$ by Lemma 3.3.6. If $|(Y - \text{cl}(X)) \cap K| = 1$, then $Y - \text{cl}(X)$ is a triad which is contained in some 4-point coline by Theorem 4.1.2. But this contradicts 5.1.9.1. So it must be that $|(Y - \text{cl}(X)) \cap K| = 2$ and the vertical 3-separation $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is balanced. Now by Lemma 5.1.7, $((\text{cl}(X) - \{b\}) \cup \{z_1\}, \{b\}, Y - (\text{cl}(X) \cup \{z_1\}))$ is a vertical 3-separation of M . If it isn't balanced, then, by Lemma 4.1.2, $Y - (\text{cl}(X) \cup \{z_1\})$ contains a triad Λ and there exists some $r \in \text{cl}(X) - \{b\}$ such that $\Lambda \cup \{r\}$ is a 4-point coline. Applying Theorem 3.4.3 with 5.1.9.1 and Lemma 3.2.4, we see that $((\text{cl}(X) - \{b, r\}) \cup \{z_1\}, \{b\}, (Y - (\text{cl}(X) \cup \{z_1\})) \cup \{r\})$ is a colinearly balanced vertical 3-separation of M .

So assume that $((\text{cl}(X) - \{b\}) \cup \{z_1\}, \{b\}, Y - (\text{cl}(X) \cup \{z_1\}))$ is balanced. If it is still not colinearly balanced, then there must exist another 4-point coline Γ_2 in M . Then $|\Gamma_2 \cap K \cap (Y - (\text{cl}(X) \cup \{z_1\}))| = 2$, so $\Gamma_2 \in \text{cl}^*(Y - (\text{cl}(X) \cup \{z_1\}))$. By 5.1.9.1, and Lemmas 3.2.4 and 3.4.3, we again find a colinearly balanced vertical 3-separation $((\text{cl}(X) - \{b\}) \cup \{z_1\} - \Gamma_2, \{b\}, (Y - (\text{cl}(X) \cup \{z_1\})) \cup \Gamma_2)$ of M .

Case (v): $|\Gamma_1 \cap Y| = 0$

Suppose $\Gamma_1 \cap Y = \emptyset$. Then $\Gamma_1 \subset X$. If $(X, \{b\}, Y)$ isn't colinearly balanced, then M contains some other 4-point coline Γ_2 . Then $|\Gamma_2 \cap Y \cap K| = 2$. Applying 5.1.9.1, Lemma 3.2.4 and Lemma 3.4.3 once more, we conclude that $(X - \Gamma_2, \{b\}, Y \cup \Gamma_2)$ is a colinearly balanced vertical 3-separation of M . \square

Corollary 5.1.10. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M . Let K denote the set of elements which are removable with respect to B , and suppose $|K| = 4$. If $(X, \{b\}, Y)$ is a colinearly balanced vertical 3-separation of M , then $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is also a colinearly balanced vertical 3-separation of M .*

Proof. Let $(X, \{b\}, Y)$ be a colinearly balanced vertical 3-separation of M . By Lemma 3.3.6, $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is a vertical 3-separation of M with $(\text{cl}(X) - \{b\}) \cup \{b\} = \text{cl}(X)$ closed. If $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is not balanced, then by Theorem 4.1.2, $Y - \text{cl}(X)$ is a triad and there exists some $r \in \text{cl}(X) - \{b\}$ such that $(Y - \text{cl}(X)) \cup \{r\}$ is a 4-point coline. But

then $r \in \text{cl}^*(Y - \text{cl}(X))$, so that $r \notin \text{cl}(\text{cl}(X) - \{r\})$. This now implies that $r(\text{cl}(X)) > r(\text{cl}(X) - \{r\})$, which contradicts the fact that $r \in \text{cl}(X)$. So $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ must be balanced. Suppose Y contains a 4-point coline Γ . Then $|(Y - \text{cl}(X)) \cap \Gamma \cap K| = 2$. As $\gamma_i \in \text{cl}^*(\Gamma - \gamma_i)$ for all $\gamma_i \in \Gamma$, it follows that $\gamma_i \notin \text{cl}(X \cup \{b\}) = \text{cl}(X)$ for all $\gamma_i \in \Gamma$ by Lemma 2.5.4. Thus $\Gamma \subseteq Y - \text{cl}(X)$ and $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is colinearly balanced. \square

Corollary 5.1.11. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M . Let K denote the set of elements which are removable with respect to B . Suppose $|K| = 4$ and there exists some $b \in B - K$. If M^* is not colinearly entwined and M is not the union of a 4-point line and a 4-point coline, then for each $c_i \in (E(M) - B) - K$ such that c_i is not on a 4-point coline, there exists a linearly balanced cyclic 3-separation $(S_i, \{c_i\}, T_i)$ of M .*

Proof. Note that a cyclic 3-separation $(S, \{c\}, T)$ of M is linearly balanced if and only if $(S, \{c\}, T)$ is a colinearly balanced vertical 3-separation of M^* . The corollary then follows by noting Lemmas 3.5.5 and 3.5.3 and dualising Proposition 5.1.9. \square

Corollary 5.1.12. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M . Let K denote the set of elements which are removable with respect to B , and suppose $|K| = 4$. If $(S, \{c\}, T)$ is a linearly balanced cyclic 3-separation of M , then $(\text{cl}^*(S) - \{c\}, \{c\}, T - \text{cl}^*(S))$ is also a linearly balanced cyclic 3-separation of M .*

Proof. This follows by dualising Corollary 5.1.10. \square

5.2 Arranging Removable Elements

Theorem 5.2.1. *Let M be a 3-connected matroid with no 4-element fans, where M is not the union of a 4-point line and a 4-point coline. Let B be a basis of M , and let K denote the set of elements which are removable with respect to B . Suppose $|K| = 4$. Then there exists a set $\{\alpha_1, \alpha_2\} \subset K$ such that for every $b_i \in B - K$, there exists a balanced vertical 3-separation $(X_i, \{b_i\}, Y_i)$ with $\{\alpha_1, \alpha_2\} \subset X_i$. Moreover, if M contains a 4-point line, then $\{\alpha_1, \alpha_2\} \subset (E(M) - B) \cap L$ where L is a 4-point line of M .*

Proof. If M is colinearly entwined, then the theorem follows immediately from Theorem 5.1.8. So assume that M is not colinearly entwined. Our argument is divided into three separate cases.

Case (i): $(E(M) - B) - K = \emptyset$.

Suppose that $(E(M) - B) - K = \emptyset$, and $b_1 \in (B - K)$. Let $(X, \{b_1\}, Y)$ be a balanced vertical 3-separation of M and let $\{\alpha_1, \alpha_2\} = X \cap K$. We show that for all other $b_i \in B - K$, there exists a balanced vertical 3-separation $(X_i, \{b_i\}, Y_i)$ with $\{\alpha_1, \alpha_2\} \subset X_i$.

Take some $b_2 \in B - K$, and assume that there does not exist any balanced vertical 3-separation $(S, \{b_2\}, T)$ with $\{\alpha_1, \alpha_2\} \subset S$. Then $b_1 \neq b_2$. Let $(P, \{b_2\}, Q)$ be a balanced vertical 3-separation of M . Without loss of generality, we may assume that $b_1 \in Q$ and $b_2 \in Y$.

5.2.1.1. For all $y \in (Y \cap K) \cup \{b_2\}$, $y \notin \text{cl}(X)$.

Proof. If $b_2 \in \text{cl}(X)$, then, by Lemma 3.2.8, $(X \cup \{b_1\}, \{b_2\}, Y - \{b_2\})$ is a vertical 3-separation that contradicts our assumption. So $b_2 \notin \text{cl}(X)$. Suppose $z \in Y \cap K \cap \text{cl}(X)$. Then by Lemma 3.3.6, $(\text{cl}(X) - \{b_1\}, \{b_1\}, Y - \text{cl}(X))$ is a vertical 3-separation of M with $|(Y - \text{cl}(X)) \cap K| \leq 1$. Now by Theorem 4.1.2, $|(Y - \text{cl}(X)) \cap K| = 1$ and $Y - \text{cl}(X)$ is a triad which contains two elements of $(E(M) - B) - K$. But $(E(M) - B) - K = \emptyset$. Therefore $Y \cap K \cap \text{cl}(X) = \emptyset$. \square

By Lemma 3.3.6, we may assume; relabelling as necessary, that $(X, \{b_1\}, Y)$ is a vertical 3-separation such that $X \cup \{b_1\}$ is closed. Moreover, by 5.2.1.1, we maintain the property that $(X, \{b_1\}, Y)$ is balanced with $b_2 \in Y$. By assumption, $\{\alpha_1, \alpha_2\} \not\subset P$ and $\{\alpha_1, \alpha_2\} \not\subset Q$. Without loss of generality, $\alpha_1 \in X \cap P$ and $\alpha_2 \in X \cap Q$. Also, $|Y \cap P \cap K| = |Y \cap Q \cap K| = 1$ because $(X, \{b_1\}, Y)$ and $(P, \{b_2\}, Q)$ are balanced. So each of the intersections $X \cap P, X \cap Q, Y \cap P$ and $Y \cap Q$ are non-empty and each contains exactly one element of K .

5.2.1.2. Either $|Y \cap P| = 1$ or $(Y \cap P) \cup \{b_2\}$ is a triangle.

Proof. Assume that $|Y \cap P| \geq 2$. As $\lambda(Y) = \lambda(P) = 2$ and $|E - (Y \cup P)| = |(X \cap Q) \cup \{b_1\}| \geq 2$, it follows by uncrossing that $\lambda(Y \cap P) = 2$. Similarly,

$\lambda((Y \cap P) \cup \{b_2\}) = 2$. By Lemmas 3.2.7 and 2.5.4, $b_2 \in \text{cl}(Y \cap P)$. If $r(Y \cap P) \geq 3$, then $(Y \cap P, \{b_2\}, X \cup Q)$ is a vertical 3-separation of M . But then by Theorem 4.1.2, $Y \cap P$ contains a triad Λ which contains elements of $E(M) - B$ which are not removable with respect to B ; contrary to our assumption. So $r(Y \cap P) = 2$. If $|Y \cap P| \geq 3$, then $(Y \cap P) \cup \{b_2\}$ contains a 4-point line L . But then $|L \cap Y \cap P \cap (E(M) - B)| \geq 2$ by Lemma 2.3.3, and so $|Y \cap P \cap K| \geq 2$ by Lemma 3.2.2, which is a contradiction. Therefore, if $|Y \cap P| \geq 2$, then $(Y \cap P) \cup \{b_2\}$ is a triangle. Otherwise, $|Y \cap P| = 1$. \square

5.2.1.3. $|X \cap P| \geq 2$.

Proof. Suppose $|X \cap P| = 1$. Then $(Y \cap P) \cup \{b_2\}$ is a triangle by 5.2.1.2 and the fact that $r(P) \geq 3$. But then P is a triad, so that $P \cup \{b_2\}$ is a contradictory 4-element fan. \square

5.2.1.4. $(Y \cap Q) \cup \{b_1, b_2\}$ is a triangle.

Proof. Noting that $\lambda(Y) = \lambda(Q) = 2$ and $|E - (Y \cup Q)| = |X \cap P| \geq 2$ by 5.2.1.3, it follows by uncrossing that $\lambda(Y \cap Q) \leq 2$. Similarly we obtain $\lambda((Y \cap Q) \cup \{b_2\}) = \lambda((Y \cap Q) \cup \{b_1, b_2\}) = 2$. Now by Lemmas 3.2.7 and 2.5.4, $b_1 \in \text{cl}((Y \cap Q) \cup \{b_2\})$. As M is 3-connected, all that remains is to show that $|Y \cap Q| = 1$. Suppose $|Y \cap Q| \geq 2$. Then $\lambda(Y \cap Q) = 2$ and it follows that $b_2 \in \text{cl}(Y \cap Q)$. If $r(Y \cap Q) \geq 3$, then $(Y \cap Q, \{b_2\}, X \cup P \cup \{b_1\})$ is a vertical 3-separation of M . But then Theorem 4.1.2 implies that $Y \cap Q$ contains a triad Λ that contains two elements of $(E(M) - B) - K$, which is absurd. So $r(Y \cap Q) = 2$. But then $r((Y \cap Q) \cup \{b_1, b_2\}) = 2$ and $(Y \cap Q) \cup \{b_1, b_2\}$ contains a 4-point line L . By Lemma 2.3.3, $Y \cap Q \subseteq E(M) - B$, and, as $|Y \cap Q \cap L| \geq 2$, we get $|Y \cap Q \cap K| \geq 2$ by Lemma 3.2.2. This is a contradiction. We conclude that $|Y \cap Q| = 1$ and that $(Y \cap Q) \cup \{b_1, b_2\}$ is a triangle. \square

Now suppose that $(Y \cap P) \cup \{b_2\}$ is a triangle. Note that Y is a rank 3 cocircuit of M . Let $Y \cap P = \{z_1, z_2\}$. Then $r(Y - \{z_1\}) = r(Y - \{z_2\}) = 3$. Therefore, both $\text{co}(M \setminus z_1)$ and $\text{co}(M \setminus z_2)$ are 3-connected by Theorem 3.4.4. As $|Y \cap P \cap K| = 1$, it follows from Lemma 2.3.3 that $|Y \cap P \cap (B - K)| = 1$. Without loss of generalization, $z_1 \in B - K$. Let $Y \cap Q = \{w\}$. Now, by Lemma 2.6.5, each of b_1, b_2 and w are contained in a distinct parallel class of M/z_1 . But this, together with the fact that $z_1 \notin K$ contradicts Lemma 3.4.1.

So $(Y \cap P) \cup \{b_2\}$ is not a triangle, and by 5.2.1.2, it must be that $|Y \cap P| = 1$. But then Y is a triad and so $Y \cup \{b_1\}$ is a contradictory 4-element fan in M . We conclude that either $\{\alpha_1, \alpha_2\} \subset P$ or $\{\alpha_1, \alpha_2\} \subset Q$ as required.

We now assume for the remaining two cases that $(E(M) - B) - K \neq \emptyset$.

Case (ii): M contains a 4-point line.

Let L be a 4-point line of M . Then $|L \cap K \cap (E(M) - B)| \geq 2$ by Lemmas 3.2.2 and 2.3.3. Choose distinct $\alpha_1, \alpha_2 \in L \cap K \cap (E(M) - B)$. We show that for each $b_i \in B - K$, there exists a balanced vertical 3-separation $(X_i, \{b_i\}, Y_i)$ such that $\{\alpha_1, \alpha_2\} \subset X_i$.

Suppose $b \in B - K$. Let $(X, \{b\}, Y)$ be a vertical 3-separation of M . Without loss of generality, we may assume that $|X \cap L| \geq 2$. By Lemma 3.3.6, $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is a vertical 3-separation of M with $\{\alpha_1, \alpha_2\} \subset L \subset \text{cl}(X) - \{b\}$. If $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is balanced, we're done. So suppose $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is not balanced. Then $|(Y - \text{cl}(X)) \cap K| = 1$, and by Theorem 4.1.2, $Y - \text{cl}(X)$ is a triad and there exists $r \in B \cap K$ such that $(Y - \text{cl}(X)) \cup \{r\}$ is a 4-point coline. So $r \in \text{cl}^*(Y - \text{cl}(X))$ and, by Lemma 2.5.4, $r \notin L$. If $r(\text{cl}(X) - \{b, r\}) = 2$, then it must be that $\text{cl}(X) - \{r\} = L$. But then M is the union of a 4-point line and a 4-point coline, which is contrary to our assumption. So $r(\text{cl}(X) - \{b, r\}) \geq 3$, and applying Lemma 3.4.3, we conclude that $(\text{cl}(X) - \{b, r\}, \{b\}, (Y - \text{cl}(X)) \cup \{r\})$ is a balanced vertical 3-separation with $\{\alpha_1, \alpha_2\} \subset \text{cl}(X) - \{b, r\}$ as required.

The argument above applies to any $b_i \in B - K$, so that if M contains a 4-point line, the theorem holds.

Case (iii): M does not contain a 4-point line.

Now assume that M does not contain any 4-point lines. As M is not colinearly entwined, we deduce from Theorem 5.1.9 that for each $b_i \in B - K$, there exists a colinearly balanced vertical 3-separation $(X_i, \{b_i\}, Y_i)$ of M . Choose some $b_1 \in B - K$ and let $(X, \{b_1\}, Y)$ be a colinearly balanced vertical 3-separation. Let $X \cap K = \{\alpha_1, \alpha_2\}$. We show that for any $b_i \in B - K$, there exists a balanced vertical 3-separation $(X_i, \{b_i\}, Y_i)$ with $\{\alpha_1, \alpha_2\} \subset X_i$.

Suppose $b_2 \in B - K$. Assume that there does not exist any balanced vertical 3-separation $(S, \{b_2\}, T)$ with $\{\alpha_1, \alpha_2\} \subset S$. Then $b_1 \neq b_2$. Let

$(P, \{b_2\}, Q)$ be a colinearly balanced vertical 3-separation of M . Without loss of generality, we may assume that $b_1 \in Q$ and $b_2 \in Y$. If $b_2 \in \text{cl}(X)$, then by Lemma 3.2.8, $(X \cup \{b_1\}, \{b_2\}, Y - \{b_2\})$ is a balanced vertical 3-separation of M that contradicts our assumption. So $b_2 \notin \text{cl}(X)$. Now by Corollary 5.1.10, we may assume; relabelling as appropriate, that $X \cup \{b_1\}$ is closed with $b_2 \in Y$. We may also assume by Corollary 5.1.10 that $Q \cup \{b_2\}$ is closed.

By assumption $\{\alpha_1, \alpha_2\} \not\subset P$ and $\{\alpha_1, \alpha_2\} \not\subset Q$. Without loss of generality, $\alpha_1 \in X \cap P$ and $\alpha_2 \in X \cap Q$. As $(X, \{b_1\}, Y)$ and $(P, \{b_2\}, Q)$ are balanced, it must also be that $|Y \cap P \cap K| = |Y \cap Q \cap K| = 1$. So each of the intersections $X \cap P, X \cap Q, Y \cap P$, and $Y \cap Q$ are non-empty and each contains exactly one element of K .

5.2.1.5. *Either $|Y \cap P| = 1$ or $(Y \cap P) \cup \{b_2\}$ is a triangle.*

Proof. Assume that $|Y \cap P| \geq 2$. As $\lambda(Y) = \lambda(P) = 2$ and $|E - (Y \cup P)| = |(X \cap Q) \cup \{b_1\}| \geq 2$, it follows by uncrossing that $\lambda(Y \cap P) = 2$. Similarly, $\lambda((Y \cap P) \cup \{b_2\}) = 2$. By Lemmas 3.2.7 and 2.5.4, $b_2 \in \text{cl}(Y \cap P)$. If $r(Y \cap P) \geq 3$, then $(Y \cap P, \{b_2\}, X \cup Q)$ is a vertical 3-separation of M . But then by Theorem 4.1.2, $Y \cap P$ contains a triad Λ and there exists some $r \in X \cup Q$ such that $\Lambda \cup \{r\}$ is a 4-point coline. But $(X, \{b_1\}, Y)$ and $(P, \{b_2\}, Q)$ are colinearly balanced so it must be that $r \in Y \cap P$, which is absurd. Hence $r(Y \cap P) \leq 2$. If $|Y \cap P| \geq 3$, then $(Y \cap P) \cup \{b_2\}$ contains a 4-point line. But M does not contain any such lines by assumption. So $|Y \cap P| \geq 2 \implies (Y \cap P) \cup \{b_2\}$ is a triangle. Otherwise, $|Y \cap P| = 1$. \square

5.2.1.6. $\lambda(Y \cap Q) \leq 2$, and $\lambda((Y \cap Q) \cup \{b_2\}) = \lambda((Y \cap Q) \cup \{b_1, b_2\}) = 2$.

Proof. Consider $|X \cap P|$. If $|X \cap P| = 1$, then $(Y \cap P) \cup \{b_2\}$ is a triangle by 5.2.1.5 and the fact that $r(P) \geq 3$. Recall that $Q \cup \{b_2\}$ is closed. But now P is a triad, and $Y \cup \{b_2\}$ is a contradictory 4-element fan in M . Therefore $|X \cap P| \geq 2$. Thus $|E - (Y \cup Q)| \geq 2$, and by uncrossing we get that $\lambda(Y \cap Q) \leq 2$. Similarly, $|E - (Y \cup (Q \cup \{b_2\}))| \geq 2$ and $|E - ((Y \cup \{b_1\}) \cup (Q \cup \{b_2\}))| \geq 2$, so that by uncrossing, $\lambda((Y \cap Q) \cup \{b_2\}) \leq 2$ and $\lambda((Y \cap Q) \cup \{b_1, b_2\}) \leq 2$. As M is 3-connected, we obtain $\lambda((Y \cap Q) \cup \{b_2\}) = \lambda((Y \cap Q) \cup \{b_1, b_2\}) = 2$. \square

5.2.1.7. $(Y \cap Q) \cup \{b_1, b_2\}$ is a triangle.

Proof. Suppose $|Y \cap Q| \geq 2$ so that $\lambda(Y \cap Q) = 2$. It then follows from 5.2.1.6 and Lemmas 3.2.7 and 2.5.4 that $\{b_1, b_2\} \subset \text{cl}(Y \cap Q)$. If $r(Y \cap Q) \geq 3$, then $(Y \cap Q, \{b_2\}, X \cup P \cup \{b_1\})$ is a vertical 3-separation of M . But $|Y \cap Q \cap K| = 1$, so that by Theorem 4.1.2, $Y \cap Q$ contains a triad Λ which is contained in a 4-point line $\Gamma \not\subset Y \cap Q$. But this is absurd as both $(X, \{b_1\}, Y)$ and $(P, \{b_2\}, Q)$ are colinearly balanced. So $r(Y \cap Q) = 2$. But then $(Y \cap Q) \cup \{b_1, b_2\}$ contains a 4-point line which cannot exist. Thus $|Y \cap Q| = 1$. By 5.2.1.6, $\lambda((Y \cap Q) \cup \{b_2\}) = \lambda((Y \cap Q) \cup \{b_1, b_2\}) = 2$, and Lemmas 3.2.7 and 2.5.4 imply that $b_1 \in \text{cl}((Y \cap Q) \cup \{b_2\})$. As M is 3-connected, $(Y \cap Q) \cup \{b_1, b_2\}$ must be a triangle. \square

Recall from 5.2.1.5 that either $|Y \cap P| = 1$ or $(Y \cap P) \cup \{b_2\}$ is a triangle. Suppose $(Y \cap P) \cup \{b_2\}$ is a triangle. Let $Y \cap P = \{z_1, z_2\}$ and $Y \cap Q = \{w\}$. Note that Y is a rank 3 cocircuit of M . Now $r(Y - \{z_1\}) = r(Y - \{z_2\}) = 3$. Therefore by Theorem 3.4.4, both $\text{co}(M \setminus z_1)$ and $\text{co}(M \setminus z_2)$ are 3-connected. But $|Y \cap P \cap K| = 1$ and $\{z_1, z_2, b_2\}$ is a triangle, so $Y \cap P$ must contain exactly one basis element. Without loss of generalization, $z_1 \in B$. But then, by Lemma 2.6.5, b_1, b_2 and w are each contained in a distinct parallel class of M/z_1 . Therefore by Lemma 3.4.1, $\text{si}(M/z_1)$ is 3-connected in addition to $\text{co}(M \setminus z_2)$ being 3-connected. This contradicts the fact that $Y \cap P$ contains exactly one element which is removable with respect to B .

The final remaining possibility is that $|Y \cap P| = 1$. But then Y is a triad, whereas $(Y \cap Q) \cup \{b_1, b_2\}$ is a triangle by 5.2.1.7. This gives a contradictory 4-element fan $Y \cup \{b_1\}$ in M . We conclude that $\{\alpha_1, \alpha_2\} \subset P$ or $\{\alpha_1, \alpha_2\} \subset Q$ as required. \square

Corollary 5.2.2. *M be a 3-connected matroid with no 4-element fans, where M is not the union of a 4-point line and a 4-point coline. Let B be a basis of M , and let K denote the set of elements which are removable with respect to B . Suppose $|K| = 4$. Then there exists a set $\{\beta_1, \beta_2\} \subset K$ such that for every $c_i \in (E(M) - B) - K$, there exists a balanced cyclic 3-separation $(S_i, \{c_i\}, T_i)$ with $\{\beta_1, \beta_2\} \subset S_i$. Moreover, if M contains a 4-point coline, then $\{\beta_1, \beta_2\} \subset B \cap \Gamma$ where Γ is a 4-point coline of M .*

Proof. Noting that $(X, \{b\}, Y)$ is a balanced vertical 3-separation of a matroid M iff it is a balanced cyclic 3-separation of M^* , this result follows by dualising Theorem 5.2.1. \square

Theorem 5.2.3. *Let M be a 3-connected matroid with no 4-element fans, where M is not the union of a 4-point line and a 4-point coline. Let B be a basis of M , and let K denote the set of elements which are removable with respect to B . Suppose $|K| = 4$. Then there exists a set $\{k_1, k_2\} \subset K$ such that:*

- (i) *for every $b_i \in B - K$, there exists a balanced vertical 3-separation $(X_i, \{b_i\}, Y_i)$ with $\{k_1, k_2\} \subset X_i$, and*
- (ii) *for every $c_i \in (E(M) - B) - K$, there exists a balanced cyclic 3-separation $(S_i, \{c_i\}, T_i)$ with $\{k_1, k_2\} \subset S_i$.*

Proof. If either $B - K = \emptyset$ or $(E(M) - B) - K = \emptyset$, then this follows immediately from the previous two results. So we can assume that both are non-empty. Our argument is divided into three cases.

Case (i): M contains a 4-point line.

Suppose M contains a 4-point line. Then by Theorem 5.2.1, there exists a 4-point line L with a set $\{\alpha_1, \alpha_2\} \subset (E(M) - B) \cap L \cap K$ such that for every $b_i \in B - K$ there exists a balanced vertical 3-separation $(X_i, \{b_i\}, Y_i)$ with $\{\alpha_1, \alpha_2\} \subset X_i$. Let $\{k_1, k_2\} = \{\alpha_1, \alpha_2\}$. Then (i) is immediate. We now prove (ii). If M^* is colinearly entwined, then (ii) follows directly from the dual of Proposition 5.1.8. So assume that M^* is not colinearly entwined. For each $c_i \in (E(M) - B) - K$ where c_i does not lie on a 4-point coline, there exists a linearly balanced cyclic 3-separation $(S_i, \{c_i\}, T_i)$ of M by Corollary 5.1.11. It follows that (ii) holds if M does not contain a 4-point coline. So let Γ be a 4-point coline of M .

Choose any $c \in (E(M) - B) - K$ and let $(S, \{c\}, T)$ be a balanced cyclic 3-separation of M . Without loss of generality, $|T \cap \Gamma| \geq 2$. By Lemma 3.3.5, $(S - \text{cl}^*(T), \{c\}, \text{cl}^*(T) - \{c\})$ is a cyclic 3-separation of M with $\Gamma \subset \text{cl}^*(T)$. Suppose $(S - \text{cl}^*(T), \{c\}, \text{cl}^*(T) - \{c\})$ is not balanced. Then by Corollary 4.1.3, $|(S - \text{cl}^*(T)) \cap K| = 1$ and $S - \text{cl}^*(T)$ is a triangle which is contained in a 4-point line. This 4-point line must be L . Without loss of generality, $(\text{cl}^*(T) - \{c\}) \cap L = \{k_1\}$. Note that $k_1 \notin \Gamma$. Now $k_1 \in \text{cl}(S - \text{cl}^*(T))$, and by Lemma 3.2.8, $\text{cl}^*(T) - \{k_1\}$ is an exactly 3-separating set. If $\text{cl}^*(T) - \{k_1\}$ does not contain a circuit, then by Lemma 3.2.6, $\text{cl}^*(T) - \{k_1\}$ is a coline. As $\Gamma \subseteq \text{cl}^*(T) - \{k_1\}$ and M contains no colines of size greater than

4 by Lemma 5.1.5, it follows that $\text{cl}^*(T) - \{k_1\} = \Gamma$. But then M is the union of a 4-point line and a 4-point coline, which is not the case. Therefore $(S - \text{cl}^*(T), \{c\}, \text{cl}^*(T) - \{c\})$ must be balanced, and $\{k_1, k_2\} \subset S - \text{cl}^*(T)$ as required. This applies to any $c_i \in (E(M) - B) - K$ so that we obtain (ii) and conclude that the theorem holds.

Case (ii): M contains a 4-point coline.

Suppose M contains a 4-point coline. Then by Theorem 5.2.2, there exists a 4-point coline Γ with a set $\{\beta_1, \beta_2\} \subset B \cap \Gamma \cap K$ such that for every $c_i \in (E(M) - B) - K$ there exists a balanced cyclic 3-separation $(S_i, \{c_i\}, T_i)$ with $\{\beta_1, \beta_2\} \subset S_i$. Let $\{k_1, k_2\} = \{\beta_1, \beta_2\}$. Then (ii) is immediate. We show (i). If M is colinearly entwined, then (ii) follows directly from Proposition 5.1.8. So assume that M is not colinearly entwined. If M contains a 4-point line, then the theorem holds as in case (i) above. So we can assume that M does not contain any 4-point lines. Then, by Corollary 5.1.9, for every $b_i \in B - K$, there exists a colinearly balanced vertical 3-separation $(X_i, \{b_i\}, Y_i)$ of M . Any such 3-separation has $\Gamma \subseteq X_i$ or $\Gamma \subseteq Y_i$ by construction, so that $\{k_1, k_2\} \subset X_i$ or $\{k_1, k_2\} \subset Y_i$ as required. Therefore (i) holds, and the Theorem follows.

Case (iii): M contains no 4-point lines and no 4-point colines.

Suppose that M does not contain any 4-point lines or 4-point colines. By Theorem 5.2.1, there exists a set $\{\alpha_1, \alpha_2\} \subset K$ such that for each $b_i \in B - K$ there exists a balanced vertical 3-separation $(X_i, \{b_i\}, Y_i)$ with $\{\alpha_1, \alpha_2\} \subset X_i$. Let $\{k_1, k_2\} = \{\alpha_1, \alpha_2\}$. Then (i) holds. We prove (ii). Take some $b \in B - K$ and let $(X, \{b\}, Y)$ be a balanced vertical 3-separation of M with $\{k_1, k_2\} \subset X$. We show that for any $c_i \in (E(M) - B) - K$, there exists a balanced cyclic 3-separation $(S_i, \{c_i\}, T_i)$ with $\{k_1, k_2\} \subset S_i$.

Suppose $c \in (E(M) - B) - K$. Assume that there does not exist any balanced cyclic 3-separation $(P, \{c\}, Q)$ with $\{k_1, k_2\} \subset P$. Let $(S, \{c\}, T)$ be a balanced cyclic 3-separation of M . Without loss of generality, we may assume that $c \in Y$ and $b \in T$

5.2.3.1. For all $y \in (Y \cap K) \cup \{c\}$, $y \notin \text{cl}(X)$.

Proof. If $c \in \text{cl}(X)$, then, by Lemma 3.2.8, $(X \cup \{b\}, \{c\}, Y - \{c\})$ is a vertical 3-separation of M . Therefore $(X \cup \{b\}, Y - \{c\})$ is a 2-separation of M/c . But $r_{M/c}(X \cup \{b\}) \geq 2$ and $r_{M/c}(Y - \{c\}) \geq 2$, so that $((E(\text{si}(M/c))) \cap (X \cup \{b\}), (E(\text{si}(M/c))) \cap (Y - \{c\}))$ is a 2-separation of $\text{si}(M/c)$. Therefore $\text{si}(M/c)$ is not 3-connected. But $c \in (E(M) - B) - K$, so that $\text{co}(M \setminus c)$ is not 3-connected. This contradicts Lemma 3.4.2. Thus $c \notin \text{cl}(X)$. If $y \in Y \cap K \cap \text{cl}(X)$, then by Lemma 3.3.6, $(\text{cl}(X) - \{b\}, \{b\}, Y - \text{cl}(X))$ is a vertical 3-separation of M . But then Theorem 4.1.2 implies that $|(Y - \text{cl}(X)) \cap K| = 1$ and $Y - \text{cl}(X)$ is contained in a 4-point coline; contrary to our assumption that M contains no such colines. So $Y \cap K \cap \text{cl}(X) = \emptyset$. \square

By Lemma 3.3.6, we may assume; relabelling as necessary, that $(X, \{b\}, Y)$ is a vertical 3-separation such that $X \cup \{b\}$ is closed. Moreover, by 5.2.3.1, we maintain the property that $(X, \{b\}, Y)$ is balanced with $c \in Y$. By assumption, $\{k_1, k_2\} \not\subset S$ and $\{k_1, k_2\} \not\subset T$. Without loss of generality, $k_1 \in X \cap S$ and $k_2 \in X \cap T$. Also, $|Y \cap S \cap K| = |Y \cap T \cap K| = 1$ because $(X, \{b\}, Y)$ and $(S, \{c\}, T)$ are balanced. So each of the intersections $X \cap S, X \cap T, Y \cap S$ and $Y \cap T$ are non-empty and each contains exactly one element of K .

5.2.3.2. $|Y \cap S| = 1$ or $Y \cap S$ is a triad.

Proof. $\lambda(Y) = \lambda(S) = 2$. Now $|E - (Y \cup S)| = |(X \cap T) \cup \{b\}| \geq 2$. So by uncrossing, $\lambda(Y \cap S) \leq 2$. Similarly, $\lambda((Y \cap S) \cup \{c\}) = 2$. Suppose $|Y \cap S| \geq 2$. Then $\lambda(Y \cap S) = \lambda((Y \cap S) \cup \{c\}) = 2$ because M is 3-connected. By Lemmas 3.2.7 and 2.5.4, $c \in \text{cl}^*(Y \cap S)$. If $Y \cap S$ contains a circuit, then, as $|Y \cap S \cap K| = 1$, applying Corollary 4.1.3, we deduce that $Y \cap S$ contains a triangle which is contained in a 4-point line. But M has no such 4-point lines. So $Y \cap S$ is independent. As $c \in \text{cl}^*(Y \cap S) - \text{cl}(Y \cap S)$, $(Y \cap S) \cup \{c\}$ is also independent. Recall that $(Y \cap S) \cup \{c\}$ is exactly 3-separating. As M does not contain any 4-point colines, Lemma 3.2.6 implies that $|Y \cap S| \leq 2$. Therefore $|Y \cap S| = 2$ and $(Y \cap S) \cup \{c\}$ is a triad. Otherwise, $|Y \cap S| = 1$. \square

5.2.3.3. $|X \cap S| \geq 2$

Proof. If $|X \cap S| = 1$, then, as $|S| \geq 3$, $(Y \cap S) \cup \{c\}$ must be a triad by 5.2.3.2. Now $(S, \{c\}, T)$ is a cyclic 3-separation of a 3-connected matroid with $|S| = 3$, so S must be a triangle. But then $S \cup \{c\}$ is a contradictory 4-element fan of M . Therefore $|X \cap S| \geq 2$. \square

5.2.3.4. $\lambda(Y \cap T) \leq 2$ and $\lambda((Y \cap T) \cup \{c\}) = \lambda((Y \cap T) \cup \{b, c\}) = 2$.

Proof. $\lambda(Y) = \lambda(T) = 2$. By 5.2.3.3, $|E - (Y \cup T)| = |X \cap S| \geq 2$. Therefore, by uncrossing, $\lambda(Y \cap T) \leq 2$. Similarly, $|E - (Y \cup (T \cup \{c\}))| = |E - ((Y \cup \{b\}) \cup (T \cup \{c\}))| = |X \cap S| \geq 2$, so by uncrossing, and the fact that M is 3-connected, $\lambda((Y \cap T) \cup \{c\}) = \lambda((Y \cap T) \cup \{b, c\}) = 2$. \square

Suppose $|Y \cap T| \geq 2$. Then by 5.2.3.4, $\lambda(Y \cap T) = \lambda((Y \cap T) \cup \{c\}) = \lambda((Y \cap T) \cup \{b, c\}) = 2$. Applying Lemmas 3.2.7 and 2.5.4, we deduce that $b \in \text{cl}(Y \cap T)$ and $c \in \text{cl}^*(Y \cap T)$. Hence, if $|Y \cap T| = 2$, $(Y \cap T) \cup \{b\}$ is a triangle. But then $((Y \cap T) \cup \{b\}, \{c\}, X \cup S)$ is a cyclic 3-separation of M . As M does not contain any 4-point lines and $|((Y \cap T) \cup \{b\}) \cap K| = 1$, this contradicts Corollary 4.1.3. So $|Y \cap T| \neq 2$. If $|Y \cap T| \geq 3$ and $Y \cap T$ contains a circuit, then we again have a cyclic 3-separation $(Y \cap T, \{c\}, X \cup S \cup \{b\})$ of M that contradicts Corollary 4.1.3. So $|Y \cap T| \geq 3 \implies Y \cap T$ is independent. But then $(Y \cap T) \cup \{c\}$ is both independent and exactly 3-separating, and Lemma 3.2.6 implies that $(Y \cap T) \cup \{c\}$ contains a 4-point coline; a contradiction.

The only remaining possibility is that $|Y \cap T| = 1$. By 5.2.3.4, $\lambda((Y \cap T) \cup \{c\}) = \lambda((Y \cap T) \cup \{b, c\}) = 2$. Therefore, by Lemmas 3.2.7 and 2.5.4, $b \in \text{cl}((Y \cap T) \cup \{c\})$. Hence $(Y \cap T) \cup \{b, c\}$ is a triangle. Now if $|Y \cap S| = 1$, then Y is a triad, but then $Y \cup \{b\}$ is a contradictory 4-element fan of M . So $(Y \cap S) \cup \{c\}$ must be a triad by 5.2.3.2. But then we find ourselves with a circuit $(Y \cap T) \cup \{b, c\}$ and a cocircuit $(Y \cap S) \cup \{c\}$ whose intersection is $\{c\}$. This contradicts Lemma 2.5.5. We conclude that either $\{k_1, k_2\} \subset S$ or $\{k_1, k_2\} \subset T$. As c was an arbitrary member of $(E(M) - B) - K$, case (iii) is complete and the theorem holds. \square

5.3 Path Width

We now tie in the results of Sections 5.1 and 5.2 with the notion of path-width (see Definition 5.0.1), and conclude with a proof of Theorem 5.0.2.

Definition 5.3.1. A *path of 3-separations* in a matroid M is an ordered partition $\mathbb{P} = (P_0, P_1, \dots, P_r)$ of $E(M)$ with the property that for all $i \in \{0, \dots, r-1\}$, $\lambda(P_0 \cup \dots \cup P_i) = 2$. The members of \mathbb{P} are called *steps* of \mathbb{P} . Empty steps are permitted.

Note that it follows from the above definition that if $\mathbb{P} = (P_0, \dots, P_r)$ is a path of 3-separations in a matroid M , then $P_0 \neq \emptyset \neq P_r$.

Note also that both vertical and cyclic 3-separations are examples of paths of 3-separations.

Lemma 5.3.2. *Let $\mathbb{P} = (P_0, \dots, P_r)$ be a path of 3-separations in a matroid M . If, for some $i \in \{1, \dots, r-1\}$, $|P_i| = 1$ with $P_i = \{e_i\}$, then either*

- (i) $e_i \in \text{cl}(P_0 \cup \dots \cup P_{i-1}) \cap \text{cl}(P_{i+1} \cup \dots \cup P_r)$, or
- (ii) $e_i \in \text{cl}^*(P_0 \cup \dots \cup P_{i-1}) \cap \text{cl}^*(P_{i+1} \cup \dots \cup P_r)$.

Proof. Suppose $P_i = \{e_i\}$ for some $i \in \{1, \dots, r-1\}$. Then $\lambda(P_0 \cup \dots \cup P_{i-1}) = \lambda(P_0 \cup \dots \cup P_{i-1} \cup \{e_i\}) = 2$. Also, $\lambda(P_{i+1} \cup \dots \cup P_r) = \lambda(P_0 \cup \dots \cup P_{i-1} \cup \{e_i\}) = 2$, and $\lambda(P_{i+1} \cup \dots \cup P_r \cup \{e_i\}) = \lambda(P_0 \cup \dots \cup P_{i-1}) = 2$. It then follows from Lemma 3.2.7 that $e_i \in \text{cl}^*(P_0 \cup \dots \cup P_{i-1})$ and $e_i \in \text{cl}^*(P_{i+1} \cup \dots \cup P_r)$. If $e_i \in \text{cl}(P_0 \cup \dots \cup P_{i-1})$, then by Lemma 2.5.4, $e_i \in \text{cl}(P_{i+1} \cup \dots \cup P_r)$; which gives (i). Otherwise, $e_i \in \text{cl}^*(P_0 \cup \dots \cup P_{i-1})$, in which case, Lemma 2.5.4 implies that $e_i \in \text{cl}^*(P_{i+1} \cup \dots \cup P_r)$; giving (ii). \square

Lemma 5.3.3. *A 3-connected matroid M has path-width 3 if and only if there is a path $\mathbb{P} = (P_0, \dots, P_r)$ of 3-separations such that $|P_0| = |P_r| = 2$, and, for all $i \in \{1, \dots, r-1\}$, $|P_i| = 1$.*

Proof. Let M be a 3-connected matroid with path-width 3. Let (e_1, \dots, e_n) be an ordering of $E(M)$ such that for all $i \in \{1, \dots, n\}$, $\lambda(\{e_1, \dots, e_i\}) \leq 2$. As M is 3-connected, it follows that for $i \in \{2, \dots, n-2\}$, $\lambda(\{e_1, \dots, e_i\}) = 2$. Now letting $P_0 = \{e_1, e_2\}$, $P_{n-3} = \{e_{n-1}, e_n\}$ and $P_i = \{e_{i+2}\}$ for $i \in \{1, \dots, n-4\}$, we obtain a suitable path of 3-separations $(P_0, P_1, \dots, P_{n-4}, P_{n-3})$.

Conversely, let $\mathbb{P} = (P_0, \dots, P_r)$ be a path of 3-separations in a 3-connected matroid M such that $|P_0| = |P_r| = 2$, and, for all $i \in \{1, \dots, r-1\}$, $|P_i| = 1$. Letting $\{e_1, e_2\} = P_0$, $\{e_{r+2}, e_{r+3}\} = P_r$ and $\{e_{i+2}\} = P_i$ for $i \in \{1, \dots, r-1\}$, we obtain an ordering (e_1, \dots, e_{r+3}) of $E(M)$ which implies that M has path-width 3. \square

Lemma 5.3.4. *Let M be a 3-connected matroid such that $E(M) = L \cup \Gamma$ where L is a 4-point line and Γ is a 4-point coline. Then M has path-width 3.*

Proof. Letting $L = \{l_1, l_2, l_3, l_4\}$ and $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, the ordering $(l_1, l_2, l_3, l_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ of $E(M)$ reveals that M has path-width 3. \square

Proposition 5.3.5. *Let $\mathbb{P} = (P_0, \dots, P_r)$ be a path of 3-separations in a matroid M . Suppose $i \in \{1, \dots, r-1\}$, $e \in P_i$, and that there exists a path of 3-separations $(X, \{e\}, Y)$ of M with $P_0 \subseteq X$, $P_r \subseteq Y$ and $e \in \text{cl}(X) \cap \text{cl}(Y)$. Then there exists a path of 3-separations $(X', \{e\}, Y')$ with $P_0 \cup \dots \cup P_{i-1} \subseteq X'$, $P_{i+1} \cup \dots \cup P_r \subseteq Y'$, and $e \in \text{cl}(X') \cap \text{cl}(Y')$.*

Proof. Let $S = P_0 \cup \dots \cup P_{i-1}$ and $T = P_{i+1} \cup \dots \cup P_r$.

5.3.5.1. *There exists a path of 3-separations $(X_0, \{e\}, Y_0)$ of M such that $S \subseteq X_0$, $P_r \subseteq Y_0$, and $e \in \text{cl}(X_0) \cap \text{cl}(Y_0)$.*

Proof. We have $\lambda(Y \cup \{e\}) = \lambda(P_i \cup T) = 2$. Note that $P_0 \subseteq X \cap S$, and P_0 contains at least two elements. Now

$$|E(M) - ((Y \cup \{e\}) \cup (P_i \cup T))| = |X \cap S| \geq |P_0| \geq 2$$

and it follows by uncrossing that $\lambda((Y \cup \{e\}) \cap (P_i \cup T)) \leq 2$. So $(X \cup S, (Y \cup \{e\}) \cap (P_i \cup T))$ is a 3-separating partition of M . We know that $|P_0| \geq 2$ and $|P_r| \geq 2$. Also, $P_0 \subseteq X \cup S$ and $P_r \subseteq (Y \cup \{e\}) \cap (P_i \cup T)$. Therefore $(X \cup S, (Y \cup \{e\}) \cap (P_i \cup T))$ is a 3-separation of M . But $e \in \text{cl}(X) \subseteq \text{cl}(X \cup S)$, and it follows from Lemmas 2.5.4 and 3.2.8 that $e \in \text{cl}(((Y \cup \{e\}) \cap (P_i \cup T)) - \{e\})$ and that $(X \cup S, \{e\}, Y \cap (P_i \cup T))$ is a path of 3-separations. Letting $X_0 = X \cup S$ and $Y_0 = Y \cap (P_i \cup T)$, the result follows. \square

Now $\lambda(X_0 \cup \{e\}) = \lambda(P_i \cup S) = 2$, and $|E(M) - ((X_0 \cup \{e\}) \cup (P_i \cup S))| = |Y_0 \cap T| \geq |P_r| \geq 2$, so that, by uncrossing, $\lambda((X_0 \cup \{e\}) \cap (P_i \cup S)) \leq 2$. As $P_0 \subseteq (X_0 \cup \{e\}) \cap (P_i \cup S)$, it must be that $\lambda((X_0 \cup \{e\}) \cap (P_i \cup S)) = 2$. Therefore $((X_0 \cup \{e\}) \cap (P_i \cup S), Y_0 \cup T)$ is an exact 3-separation of M . As $e \in \text{cl}(Y_0 \cup T)$, it then follows from Lemmas 2.5.4 and 3.2.8 that $e \in \text{cl}(((X_0 \cup \{e\}) \cap (P_i \cup S)) - \{e\})$ and that $(X_0 \cap (P_i \cup S), \{e\}, Y_0 \cup T)$ is a path of 3-separations. Observing that $S \subseteq X_0 \cap (P_i \cup S)$ and $T \subseteq Y_0 \cup T$, we conclude that the theorem holds. \square

Corollary 5.3.6. *Let $\mathbb{P} = (P_0, \dots, P_r)$ be a path of 3-separations in a matroid M . Suppose $i \in \{1, \dots, r-1\}$, $e \in P_i$, and that there exists a path of 3-separations $(X, \{e\}, Y)$ in M with $P_0 \subseteq X$, $P_r \subseteq Y$ and $e \in \text{cl}^*(X) \cap \text{cl}^*(Y)$. Then there exists a path of 3-separations $(X', \{e\}, Y')$ with $P_0 \cup \dots \cup P_{i-1} \subseteq X'$, $P_{i+1} \cup \dots \cup P_r \subseteq Y'$, and $e \in \text{cl}^*(X') \cap \text{cl}^*(Y')$.*

Proof. This follows by duality from Proposition 5.3.5. \square

Corollary 5.3.7. *Let $\mathbb{P} = (P_0, \dots, P_r)$ be a path of 3-separations in a matroid M . Suppose $i \in \{1, \dots, r-1\}$, $e \in P_i$, and that there exists a path of 3-separations $(X, \{e\}, Y)$ in M with $P_0 \subseteq X$ and $P_r \subseteq Y$. Then there exists a path of 3-separations $(X', \{e\}, Y')$ with $P_0 \cup \dots \cup P_{i-1} \subseteq X'$ and $P_{i+1} \cup \dots \cup P_r \subseteq Y'$.*

Proof. This is a consequence of Lemma 5.3.2, together with the results of Proposition 5.3.5 and Corollary 5.3.6. \square

Corollary 5.3.8. *Let $\mathbb{P} = (P_0, \dots, P_r)$ be a path of 3-separations in a matroid M . Suppose $i \in \{1, \dots, r-1\}$, $e \in P_i$, and that there exists a path of 3-separations $(X, \{e\}, Y)$ of M with $P_0 \subseteq X$ and $P_r \subseteq Y$. Then \mathbb{P} refines to a path $(P_0, \dots, P_{i-1}, P'_i, \{e\}, P''_i, P_{i+1}, \dots, P_r)$ of 3-separations where $P'_i \cup \{e\} \cup P''_i = P_i$.*

Proof. By Corollary 5.3.7, we have a path of 3-separations $(X', \{e\}, Y')$ of M with $(P_0 \cup \dots \cup P_{i-1}) \subseteq X'$ and $(P_{i+1} \cup \dots \cup P_r) \subseteq Y'$. Letting $P'_i = X' - (P_0 \cup \dots \cup P_{i-1})$ and $P''_i = Y' - (P_{i+1} \cup \dots \cup P_r)$, it follows that $(P_0, \dots, P_{i-1}, P'_i, \{e\}, P''_i, P_{i+1}, \dots, P_r)$ is a path of 3-separations. Moreover, $P'_i \cup \{e\} \cup P''_i = P_i$. \square

Corollary 5.3.9. *Let M be a 3-connected matroid with disjoint sets $\{a_1, a_2\} \subset E(M)$ and $\{z_1, z_2\} \subset E(M)$. Suppose that for every $e \in E(M) - \{a_1, a_2, z_1, z_2\}$, there exists a path of 3-separations $(X, \{e\}, Y)$ of M such that $\{a_1, a_2\} \subseteq X$ and $\{z_1, z_2\} \subseteq Y$. Then M has path-width 3.*

Proof. As M is 3-connected, $(\{a_1, a_2\}, E(M) - \{a_1, a_2, z_1, z_2\}, \{z_1, z_2\})$ is a path of 3-separations. If $E(M) = \{a_1, a_2, z_1, z_2\}$, the result is immediate. So suppose that $e \in E(M) - \{a_1, a_2, z_1, z_2\}$. Applying Corollary 5.3.8, we obtain a refinement of our original path of 3-separations. Now successively applying Corollary 5.3.8 to each of the other elements of $E(M) - \{a_1, a_2, z_1, z_2\}$; each time with respect to our new refined path of 3-separations, we eventually obtain a path of 3-separations $\mathbb{P} = (\{a_1, a_2\}, P_1, \dots, P_q, \{z_1, z_2\})$ where, for all $i \in \{1, \dots, q\}$, the step P_i is either a singleton, or empty. If we then remove all empty steps from \mathbb{P} , we obtain a path of 3-separations $\mathbb{P}' = (\{a_1, a_2\}, P'_1, \dots, P'_n, \{z_1, z_2\})$ in which P_i is a singleton for all $i \in \{1, \dots, n\}$. By Lemma 5.3.3, M has path-width 3. \square

Proof of Theorem 5.0.2. If M is the union of a 4-point line and a 4-point coline, then the result follows immediately from Lemma 5.3.4. So assume that this is not the case. Let $K = \{k_1, k_2, k_3, k_4\}$. Recall that vertical 3-separations and cyclic 3-separations are both examples of paths of 3-separations. Therefore it follows from Theorem 5.2.3 that for each $e_i \in E(M) - K$, there exists a path of 3-separations $(X_i, \{e_i\}, Y_i)$ of M such that $\{k_1, k_2\} \subset X_i$ and $\{k_3, k_4\} \subset Y_i$. Thus, by Corollary 5.3.9, M has path-width 3. \square

Chapter 6

Future Work

6.1 Some Conjectures

Our motivation for wanting to maintain 3-connectivity upon removing elements from matroids arose primarily from our desire to obtain matroids that are uniquely representable. But as discussed in Section 2.7, in general, for a matroid to have any sense of unique representability over $GF(q)$, not only must that matroid be 3-connected, but we also require that it contain a stabilizer for $GF(q)$ as a minor. While we have shown that for a suitable matroid M with arbitrary basis B , there always exists at least four elements which are removable with respect to B ; thereby maintaining 3-connectivity, in principle, it is possible that removing any one of these elements means losing the existence of a stabilizer as a minor; hence losing our sense of unique representability. However, we believe that something can be said provided there exists *some* element of $E(M)$ that can be deleted or contracted as appropriate while maintaining the stabilizer as a minor. We make the following conjecture, which we believe should follow from results established in this thesis.

Conjecture 6.1.1. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M . Let N be a 3-connected minor of M . If there exists some $b \in B$ such that M/b has an N -minor, or there exists some $c \in E(M) - B$ such that $M \setminus c$ has an N -minor, then there exists distinct $k_1, k_2 \in E(M)$ such that for $i \in \{1, 2\}$,*

- (i) $\text{si}(M/k_i)$ is 3-connected with an N -minor if $k_i \in B$, and
- (ii) $\text{co}(M \setminus k_i)$ is 3-connected with an N -minor if $k_i \in E(M) - B$.

We showed in Chapter 5 that for a suitable matroid M and basis B , if there are *exactly* four elements which are removable with respect to B , then M necessarily has path-width 3. Now the class of matroids with path-width 3 is well understood (see [1],[6]). Therefore, one should be able to give a stronger, more explicit, description of the class of matroids which have minimal removability with respect to some basis. For the moment, we leave this as an open problem.

Also, assuming that Conjecture 6.1.1 holds, it would be of interest to study the structure that is imposed upon M if there are *exactly* two elements which we can remove with respect to B while maintaining an N -minor. Furthermore, it is plausible that from a confirmation of Conjecture 6.1.1, we may be able to strengthen the result by removing the restriction that we need to simplify and cosimplify to remain 3-connected. Indeed, Oxley et al [10] showed the following:

Theorem 6.1.2. *Let M be a 3-connected matroid with no 4-element fans, and let N be a 3-connected minor of M . Let B be a basis of M , and assume that either there is an element b_1 of B such that M/b_1 has an N -minor, or there is an element b_1^* of $E(M) - B$ such that $M \setminus b_1^*$ has an N -minor. Then either:*

- (i) B contains an element b such that M/b is 3-connected with an N -minor, or
- (ii) $E(M) - B$ contains an element b^* such that $M \setminus b^*$ is 3-connected with an N -minor.

We next propose such a strengthening of Conjecture 6.1.1; analogous to Theorem 6.1.2. However it seems quite possible that in this case, extra constraints on the matroid M may be required.

Conjecture 6.1.3. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M . Let N be a 3-connected minor of M . If there exists some $b \in B$ such that M/b has an N -minor, or there exists some $c \in E(M) - B$ such that $M \setminus c$ has an N -minor, then there exists distinct $k_1, k_2 \in E(M)$ such that for $i \in \{1, 2\}$,*

- (i) M/k_i is 3-connected with an N -minor if $k_i \in B$, and
- (ii) $M \setminus k_i$ is 3-connected with an N -minor if $k_i \in E(M) - B$.

In the event that Conjecture 6.1.3 was shown to hold, we could then take N to be the empty matroid in 6.1.3 and obtain the following as a corollary.

Conjecture 6.1.4. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M . Then there exists distinct $k_1, k_2 \in E(M)$ such that for $i \in \{1, 2\}$,*

- (i) M/k_i is 3-connected if $k_i \in B$, and
- (ii) $M \setminus k_i$ is 3-connected if $k_i \in E(M) - B$.

Finally, we note an observation on Theorem 5.0.2. We showed that if a 3-connected matroid M with no 4-element fans, and *some given* basis B contains exactly four elements which are removable with respect to B , then the matroid M *itself* has path-width 3. But the notion of removability is relative to the choice of basis one makes. It is certainly possible that upon switching to another basis B' , we find that M contains more than four removable elements. Potentially, one could examine the *entire collection* of bases of suitable matroids. Perhaps one could deduce that, for any 3-connected matroid M with no 4-element fans, *there exists* some basis B^\exists such that M contains more than four elements which are removable with respect to B^\exists .

Question 6.1.5. Does there exist a 3-connected matroid M with no 4-element fans such that for every basis B of M , there exists exactly four elements of $E(M)$ which are removable with respect to B ? If so, what additional structure must such an M have?

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