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VIABILITY OF AN ECONOMY WITH CONSTRAINED INEQUALITY

JACEK B. KRAWCZYK & WILBUR TOWNSEND

Abstract. Governments want to prevent high inequality while maintaining economic efficiency. This paper investigates how an economy can satisfy both these constraints. We use the relative factor share as a proxy for inequality and so can use a representative agent model to understand how inequality evolves. Our representative agent model includes capital, consumption and debt which, like the relative factor share, are influenced by tax rates. Whether the model's evolutions can be constrained is understood as a problem of viability theory, and so we compute the viability kernels corresponding to our constraints. These kernels explain both how policy makers should act and why they act as they currently do. For example, we show that substantial government debt will require policy makers to reduce inequality. More importantly, we demonstrate that viability theory is a meaningful, interesting approach to understanding the tradeoff between inequality and efficiency.

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Contents

1. Introduction

In his paradigmatic 'Equality and Efficiency: The Big Tradeoff', Okun (1975) describes egalitarian policies as like leaking buckets: governments can transfer wealth to reduce inequality, but when they do so some wealth will be lost. While this tradeoff between inequality and economic efficiency has produced a rich macroeconomic literature, an explicit framework for understanding the policy problem the tradeoff produces has not yet been produced. Governments will have opinions about the acceptable levels of efficiency and inequality, and the ultimate policy problem is whether these opinions are consistent – whether there are any economic states that are compatible with constraints on both efficiency and inequality. That is this problem which, in this paper, we begin to solve.

We solve this problem with viability theory, the mathematical theory of constrained dynamic systems. Viability theory determines a set of initial conditions – the viability kernel – from which a dynamic system can be controlled within constraints. Viability theory has found many applications in economics, but this is its first application to income inequality. Section 2 explains viability theory and how it conceives policy problems.

We derived our economic model in Krawczyk and Townsend (2015a). The focus of this model is the "relative factor share" – the ratio of net capital income to net labour income. We know from Krawczyk and Townsend (2015b) that the relative factor share is highly correlated with income inequality, particularly the income inequality between a society's most wealthy and its masses. The relative factor share is an economic aggregate, and by focusing on an economic aggregate we allow ourselves to understand inequality through a representative agent model. While we constrain inequality through constraining the relative factor share, we constrain efficiency by constraining consumption, capital stocks and debt (inefficient economies discourage capital formation, restrict consumption and increase government debt). The model, its parametrisation and its interpretation in viability theory are explained in Section 3. [1]

The remainder of the paper discusses the results of our viability analysis. Section 4 discusses the viability kernels we produce. Section 5 compares the trends that our simulations follow as they are controlled within our constraints. Section 6 asks how an economy can reduce its relative factor share while remaining within other constraints. In an appendix we explain the numerical method which computed our kernels.

This paper produces a framework for understanding when Okun's bucket matters – when the tradeoff between inequality and efficiency makes an economic state unacceptable. That framework yields useful results, demonstrating that it deserves to be at the core of inequality research in the future.

2. Viability theory

2.1. **The meaning of viability.** Viability theory is the mathematics which studies constrained dynamic systems. A system's evolution is *viable* if the system remains, for the entire time of the evolution, within a constraint set K. The *viability kernel* is a subset of K which contains all points which can be made viable, given a constrained control set U.

[1]Our model describes an economy over the medium term: it explains changing levels of capital and debt, but not changing technology. Thus our results cannot be compared to those from longer term studies, such as the Kuznets curve of Kuznets (1955).

Viability theory attempts to determine whether a nonempty viability kernel exists and, if so, what its boundaries are.

Viability theory formalises the 'satisficing' policies of Simon (1955) – so long as viability is not threatened, any policy is good enough. This characterisation provides a good description of real world decision-making. For example, an inflation-targeting central banker will avoid changing interest rates until doing so is necessary. This will be more naturally expressed as a viability problem than as an optimisation problem, and management theories based on viability will be closer to how managers actually behave.

Most viability theory applications have focused on environmental policy – see for example Béné, Doyen and Gabay (2001), Martinet and Doyen (2007), De Lara, Doyen, Guilbaud and Rochet (2006), and Martinet, Thébaud and Doyen (2007). Viability theory has also been applied to finance (see Pujal and Saint-Pierre (2006)), managerial economics (see Krawczyk, Sissons and Vincent (2012)), macroeconomics (see Clément-Pitiot and Saint-Pierre (2006), Clément-Pitiot and Doyen (1999), Krawczyk and Kim (2009), Krawczyk and Kim (2014), Bonneuil and Saint-Pierre (2008), Bonneuil and Boucekkine (2008), Krawczyk and Kim (2004), Krawczyk and Sethi (2007)) and microeconomics (see Krawczyk and Serea (2013)).

To illustrate viability we reproduce Figure 1 from Krawczyk and Pharo (2013). The state constraint set K is represented by the yellow (or light shadowed) shape contained in state space X. The solid and dashed lines symbolise system evolutions, which converge to where the arrows end. The brown (darker shadowed) shape is the viability kernel. The trajectories that start in the kernel remain in K and are thus viable. The trajectories that start outside the kernel will eventually leave K.

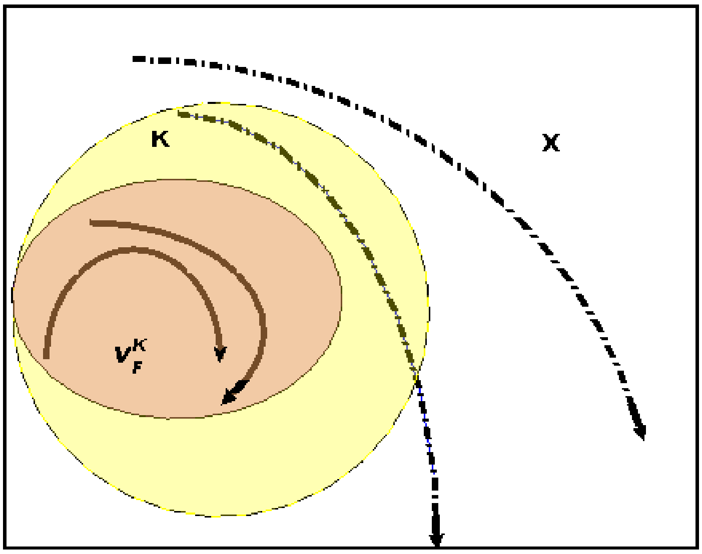

FIGURE 1. The viable and non viable trajectories for a time-invariant dynamic system.

2.2. **The mathematical formulation of viability.** Rigorous introductions to viability theory can be found in Aubin (1991), Quincampoix and Veliov (1998), Veliov (1993) and Aubin, Bayen and Saint-Pierre (2011). Here we present only these notions of viability theory which are essential to our analysis.

In viability theory, the differential inclusion

$$\dot{x}(t) \in F(x(t)) \tag{1}$$

is the basic description of a dynamic system. It states that at $x(t)$ the change in the system's state – its velocity – will be a member of $F(x(t))$, where F is a set-valued map from system states to sets of possible velocities. In control theory the map F has the form $F(x) = f(x, U) = \{f(x, u); u \in U\}$, where $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a continuous vector-valued function representing the system's equations of motion and U is a compact set in $\mathbb{R}^m$.[2] In this case, we can re-write (1) as

$$\dot{x}(t) = f(x(t), u(t)) \tag{2}$$

$$u(t) \in U(x(t)) \tag{3}$$

where (2) is a standard parameterised differential (vector) equation and (3) states that the control choice $u(\cdot)$ must come from a set $U(x(\cdot))$, which may be state-dependent.

As above, let K represent the closed set of constraints that state $x(t)$ must satisfy for all t – say, an inflation and output-gap constraint. Given a set-valued map $F : K \rightsquigarrow \mathbb{R}^n$, we say that $x_0 \in K$ is *viable in K under F* if there exists at least one solution to the following system:

$$\forall t \in \Theta \begin{cases} x(t) \in K, \\ \dot{x}(t) \in F(x(t)), \end{cases} \tag{4}$$

that starts at $x(0) = x_0$ and remains in K forever: $\Theta \equiv [0, \infty)$.[3]

Formulation (4) describes the viability of an individual system state. The viability kernel $\mathcal{V}_F(K)$ is the set of all viable states:

$$\mathcal{V}_F(K) \equiv \{x(0) : \exists\, x(t) \,\text{satisfying (2)-(3) and constraints}\; K\, \forall t\}. \tag{5}$$

For a control problem, the viability kernel $\mathcal{V}_F(K)$ is the area in which a control exists which can keep the system within K indefinitely. If a trajectory begins inside the viability kernel $\mathcal{V}_F(K)$ then we have sufficient controls to keep this trajectory in the constraint set K for all t. If a trajectory begins outside the kernel then it will inevitably leave K. The viability kernel $\mathcal{V}_F(K)$ has important implications for policy. In particular, it allows us to construct control rules that maintain the system's viability.

[2]For other interpretations of (1) see Krawczyk and Pharo (2013).

[3]Viability is normally defined in terms of an infinite time horizon, but this is not necessary.

3. Formulating the viability of constrained relative factor share

We will use the same notation as in Krawczyk and Townsend (2015a) and Krawczyk and Townsend (2015b). In particular, χ is the relative factor share – capital income, less depreciation, divided by labour income. As shown in Krawczyk and Townsend (2015b), χ correlates with the shares of income taken by the highest income 1% and 0.1%. So, inequality will diminish in line with the relative factor share.

3.1. **The viability kernel.**

We have derived in Krawczyk and Townsend (2015a) formulae for the relative factor shares for one tax and two tax economies. Respectively, they are:

$$\chi \equiv \frac{k\bar{r}}{l\bar{w}} = \frac{\alpha}{1-\alpha} - \delta\left(\left(\frac{Vc^{\gamma}}{1-\tau}\right)^{1-\alpha} k^{\eta(1-\alpha)}(A(1-\alpha))^{-(\eta+1)}\right)^{\frac{1}{\alpha+\eta}} \tag{6}$$

and

$$\chi \equiv \frac{k\bar{r}}{l\bar{w}} = \frac{1-\tau_K}{1-\tau_L}\left(\frac{\alpha}{1-\alpha} - \delta\left(\left(\frac{Vc^{\gamma}}{1-\tau_L}\right)^{1-\alpha} k^{\eta(1-\alpha)}(A(1-\alpha))^{-(\eta+1)}\right)^{\frac{1}{\alpha+\eta}}\right). \tag{7}$$

Let $x(t)$ be the state vector composed of capital k, consumption c, debt B and taxation rate τ[4]. We ask whether the system dynamics $F(x(t))$

[4] In the remainder of this paper we will assume that $\tau_K = \tau_L = \tau$ where $\tau \geq 0$ is *income* taxation rate.

are compatible with the viability constraints K:

$$K \equiv \left\{ (k, c, B, \tau) : \left| \begin{array}{c} \underline{k} \leq k(t) \ \leq \overline{k} \\ \\ \underline{c} \leq c(t) \ \leq \bar{c} \\ \\ \underline{B} \leq B(t) \ \leq \overline{B} \\ \\ \tau(t) \ \in \ [\tau_{\min}, \tau_{\max}] \\ \\ 0 \leq \chi \leq \overline{\chi} \end{array} \right. \right\}. \tag{8}$$

where the constraints on k, c, B, τ, χ – $\underline{k}, \overline{k}, \underline{c}$, *etc.* – will be explained in the next sub-section.

If the system's dynamics are compatible with K, there will exist a set of economic states from which there exist viable evolutions that respect the entire set of constraints. This is the viability kernel discussed earlier, here given as

(9)

$$\mathcal{V}_F(K) \equiv \left\{ (k(0), c(0), B(0), \tau(0)) : \left| \begin{array}{c} \exists \ (k(\cdot), c(\cdot), B(\cdot), \tau(\cdot)), \\ \text{starting from } (k(0), c(0), B(0), \tau(0))) \\ \text{satisfying dynamics } F(x(t)), \\ u \in U \text{ and constraints (8)} \\ \forall \, t \ \in \ \Theta \end{array} \right. \right\}.$$

where U contains allowable taxation-rate *adjustments* (perhaps $\pm$ 20% per year).

A regulator of the economy described by the dynamics $F(x(t))$ and the constraint set K will be seeking strategies $u(\cdot)$ that generate $k(\cdot), c(\cdot), B(\cdot), \tau(\cdot)$ consistent with the above definition of $\mathcal{V}_F(K)$.

3.2. **The calibration.** Following Krawczyk and Judd (2015), this paper analyses kernels produced for a "reasonably industrialized economy composed of rational agents interested in the near future, drawing a fair satisfaction from consumption and feeling, quite strongly, the burden of labor". We assume $\rho = 0.04$, $\alpha = 0.43$, $\eta = 1$ and $\gamma = 0.5$. In contrast to Krawczyk and Judd (2015) where $\delta = 0$, we assume $\delta = 0.05$. When $\delta = 0$ tax has no impact on χ, see (6).

Using a stylised steady state $\underline{k} = \underline{\ell} = 1$ with no taxes and no government expenditure, we calibrate A and V and obtain $A = 0.2093$, $V = 0.2989$. We then assume that government expenditure g is constant and set at 10% of no-tax steady-state output; $g = 0.1 \cdot A = 0.0209$.

The constraints come from a combination of positive and normative sources, as well as from the requirement to close K. For example, the lower bound on capital might be tied to a normative requirement concerning the nation's GDP, whereas the upper bound might be based only on the observation that capital would never realistically fluctuate that far from its steady state.

(I) **Capital** should be within 10% and 200% of no-tax steady state capital stock, $k \in [0.1,\ 2]$;

(II) **consumption** should not deviate too far from a long-run equilibrium (see Krawczyk and Judd (2015)), $c \in [0.0267,\ 0.225]$;

(III) **debt** may grow to 150% of the maximum steady-state capital stock and also drop somewhat below zero, $B \in [-1,\ 3.5]$;

(IV) **tax rate** cannot be less than zero, and can at most be equal to 80%, $\tau \in [0.\ 0.8]$;

(V) **tax-rate adjustment speed** – the amount the regulator increases or decreases the tax rate within a year – will be less than 20 percentage points, $u \in [-0.2,\ 0.2]$.

We will first require χ to be ≤ 0.4, and then next require it to be ≤ 0.25. The relative factor share $\chi = 0.4$ corresponds per our FGLS analysis in Krawczyk and Townsend (2015b) to the top 1% taking 4.2% of income and to the top 0.1% taking 0.97% of income. $\chi = 0.25$ corresponds to the top 1% taking 3.5% and to the top 0.1% taking 0.76%. (In New Zealand the top 1% currently take about 8% of national income. In the mid-1980s they were taking about 5.5%. See Krawczyk and Townsend (2015b) for more detail.)

We also require χ to be positive. This is less a normative constraint, more an interpretation aid. Viability theory finds points from which a system can be kept within certain bounds. (This does not require that the system be kept at those points.) Negative χ requires negative interest rates, as wages (the marginal product of labour) will be positive in a Cobb-Douglas production function. Negative interest rates require $\frac{\partial Y}{\partial k} < \delta$. Our simulations confirm that such a situation is not sustainable: capitalists will not invest if they are receiving a negative return. Thus requiring $\chi > 0$ removes points which are viable but not the result of any long-run steady state, simplifying our analysis.

Thus the constraint set K for one tax, for which we will find the viability kernel, is

$$(10) \quad K = [0.1,\, 2] \times [0.0267,\, 0.225] \times [-1,\, 3.5] \times [0,\, 0.8] \times [0,\, \overline{\chi}]\,,$$

where $\overline{\chi}$ is either 0.4 or 0.25.

4. Viability kernel comparison

In this section we analyse the viability of different relative factor share constraints. We want to see if there are feasible tax-rate adjustment strategies that lead to economies with a constrained relative factor share level while capital, consumption and debt are kept within some bounds. This requires finding the viability kernel $\mathcal{V}_F(K) \in K \subset \mathbb{R}^4$ for the dynamics $F(x(t))$. We will use VIKAASA, a piece of specialised software briefly introduced in Appendix A, to compute $\mathcal{V}$. As said before, we limit our attention in this paper to the case when capital and labour are taxed at the same rate, τ. (By definition, viability kernels can only become larger with two tax rates.)

Figure 2 shows 3D kernel slices for the kernels produced by three different sets of constraints. The kernels include points regardless of their initial debt level, whereas those in Figures 3 and 5 require debt to start at some level.

As discussed above, all three slices require $\chi \geq 0$. The first has no further restrictions. The second requires $\chi \leq 0.4$. The third requires $\chi \leq 0.25$. These slices are projected onto the consumption-capital axis and shown in black.

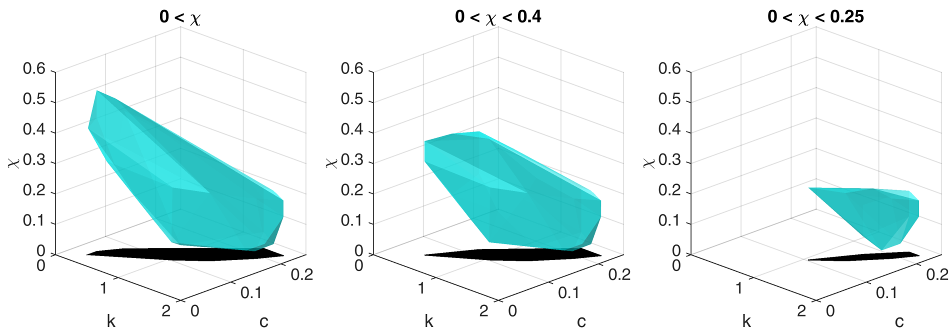

FIGURE 2. Viability kernels for different relative factor share constraints

Both the kernels and their projections shrink as the constraint is imposed. In particular, low levels of capital become non-viable when the relative factor share is required to be less than 0.4. However, this effect is small. There is a much larger reduction in the kernel when the constraint is lowered to $\chi < 0.25$. At this point, only high levels of capital are viable.

We will now consider a question more relevant for policy: whether low inequality targets remain viable when a government has significant debt. In Figure 2 states were included regardless of their initial government debt, provided that government debt could be controlled to remain in $[-1, 3.5]$. Figure 3 requires debt to start equal to 3.5.

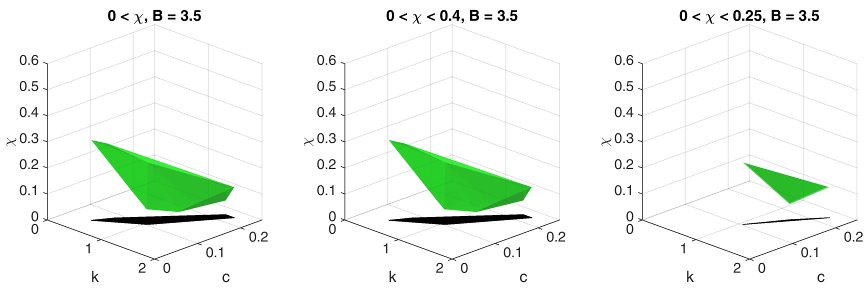

FIGURE 3. Viability kernels with high debt

The high-debt kernels are noticeably different to those in Figure 2. In particular, a high relative factor share (and thus high inequality) is impossible with high levels of government debt. The left-most panel has no constraint on the relative factor share, but nonetheless the factor share is always < 0.4.

To see why high debt prevents a high relative factor share and – by extension – prevents high inequality, note that the relative factor share isn't (directly) a function of debt. However, the relative factor share is decreasing in tax rates, suggesting high tax rates reduce capital income more than they reduce labour income (see (6)). Thus if high debt requires high tax rates then the corresponding relative factor share will be low. To demonstrate this, consider the converse case of low tax rates. Low tax rates will lead to a high χ, but are only viable with low initial debt. Set tax $= 0.2$, capital $= 0.2$ and consumption $= 0.02$. This corresponds to $\chi = 0.3549$. We impose only a bottom constraint $\chi \geq 0$ and compare the trajectories which begin of debt $= 0$ – to debt

= 3. We use VIKAASA to see which taxation strategy can keep the economy within the constraints on B, c, k and τ.

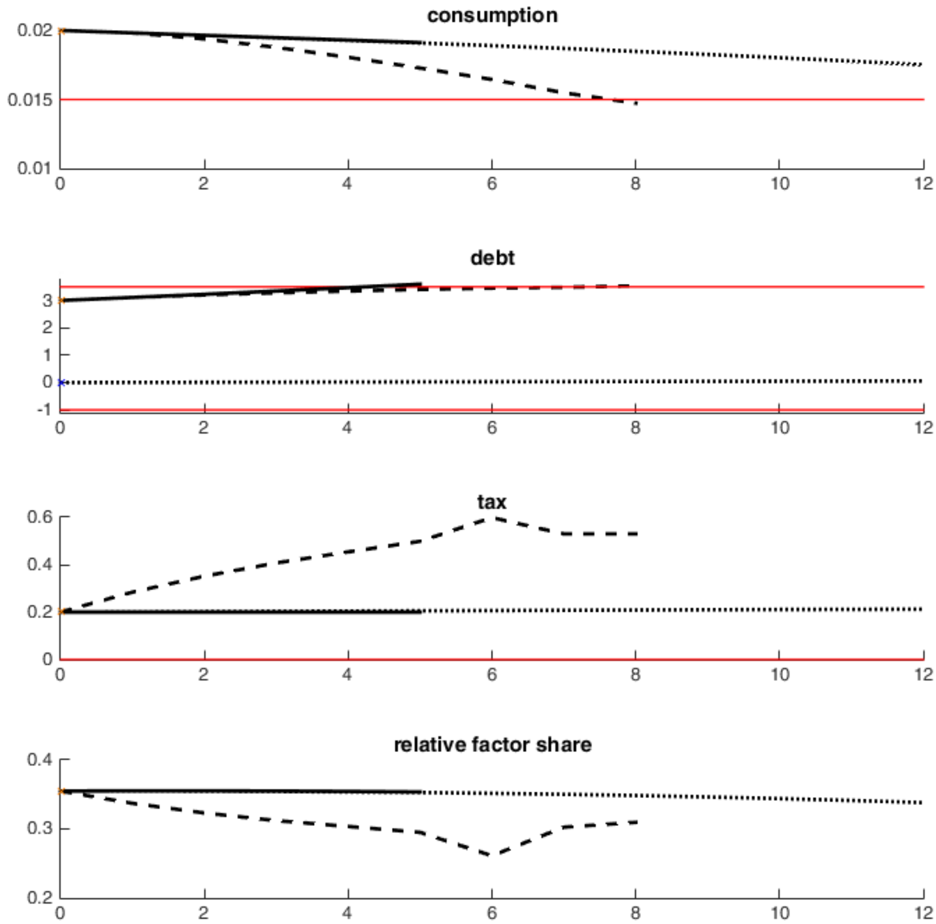

FIGURE 4. Time profiles of high debt (solid lines without tax changes, dashed with tax changes) and low debt (dots)

Figure 4 shows the time profiles of consumption, debt, tax and the relative factor share. If debt is high (dashed lines) we must increase taxes, otherwise the debt will crash through the upper bound. But, doing so at full speed (to keep debt < 3.5) pushes consumption below its minimum constraint, showing us that the initial point is not viable. When we do not increase taxes (solid line) we go above the debt constraint. However, when we start from zero debt (dotted lines), debt can be stabilised without increasing taxes, allowing us to also stabilise consumption. We can see that high inequality is viable with low debt, but not with high debt.

The above analysis leads us to the following conclusion: *highly indebted economies will have neither high inequality nor low tax rates.*

This does indeed seem to be the case. Japan has the highest public debt in the world, with public debt in 2010 equalling 206% of GDP (The World Bank, 2015). Our model predicts that Japan will have neither low tax rates nor high inequality: this is correct, Japan's 1% share in 2010 was 9.51% (Alvaredo, Atkinson, Piketty and Saez, 2014) and its top marginal tax rate was 40% (National Tax Agency, 2010). In contrast the country in the Alvaredo et al. (2014) database with the highest 1% share in 2010 was Columbia, where the wealthiest 1% take 20.45% of national income. Columbia had debt equal to only 38% of GDP (The World Bank, 2015).

The above analysis considered high-debt economies. In contrast, Figure 5 includes only viable points which start with debt equal to 0. The kernel which required $\chi < 0.25$ had too few viable points to generate a three-dimensional slice, and is thus excluded.

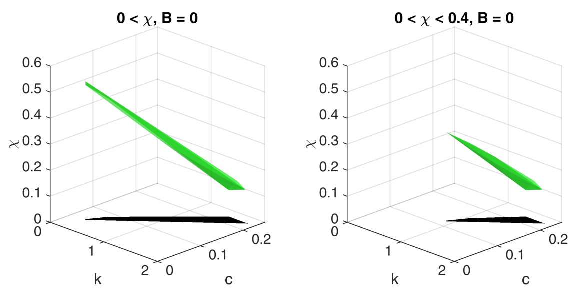

FIGURE 5. Viability kernels with debt = 0

The low debt kernels, both unconstrained and with $\chi \leq 0.4$, have much fewer viable points than their high debt counterparts in Figure 3. This is surprising: intuitively, lower debt would give governments more flexibility. The low-debt kernels are so small because of the bottom constraint on debt, $B \geq -1$. This constraint can be justified – perhaps we are concerned about the political-economic implications of an economy in which governments control all capital, perhaps we are concerned about the macroeconomic impact of a savings glut. In any case, the bottom constraint is needed by VIKAASA which requires a compact constraint set. In summary, a low debt economy with high taxation rates and high capital would quickly accumulate excess savings.

5. Stabilising paths

Section 4 discusses viability kernels, the states from which an economy can be controlled to remain within our constraints K indefinitely. These states are not necessarily stable – many will have to be controlled with changing tax rates to remain in K. That will change their χ. While the constrained kernels demonstrate that some states can be controlled while retaining low χ, it is unclear whether χ typically converges as an economy is stabilised.

VIKAASA confirms viability of a point by finding a path emanating from that point that leads the economy to a near-steady state. These paths are not unique, and strategies that generate these paths are not the only viable strategies. Nevertheless it is interesting to examine the stabilising paths and investigate their patterns.

We first compare the 'final' values of χ with those 'initial' values shown in Figure 2. Figure 6 depicts the distributions of χ across all viable states. The first panel depicts these states' initial χ, the second depicts their χ once they have been stabilised. Both panels take states from the $\chi \geq 0$ kernel, and allow any viable value of B.

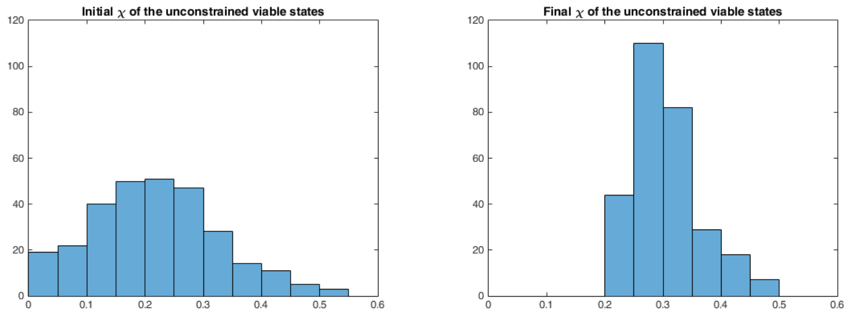

FIGURE 6. Final and initial χ distributions

As can be seen, the stabilised states of χ are less widely distributed, and in particular there are no stable states with extremely low inequality. If the stable states which correspond to the paths VIKAASA has found are representative of all possible stable states then extremely low inequality is unsustainable.

The next figures[5] show a sample of time profiles of viable evolutions for unconstrained χ – Figure 7 – and $\chi \leq 0.4$ – Figure 8. The selected sample paths are those that start far from a near-steady state. They are sufficiently long for us to see how near-steadiness is achieved .

Strikingly, the figures differ very little. This is partially because we are examining only viable states, and we know from Figure 6 that viable states converge to χ near 0.4 even when no top constraint on χ is imposed. But this is only a partial explanation: these trajectories' initial states correspond to points in the kernels of Figure 2. Those kernels shrunk substantially as the top constraint on χ was introduced, but that shrinkage is not obvious comparing the initial points of Figure 7 to those of Figure 8.

[5]The graph names are self explaining apart from *velocity* which is the Euclidean norm of each state variable velocity at a point, $\sqrt{\dot{k}^2 + \dot{c} + \dot{B} + \dot{\tau}^2}$. Near-steadiness is achieved when this norm is less than a tolerance parameter.

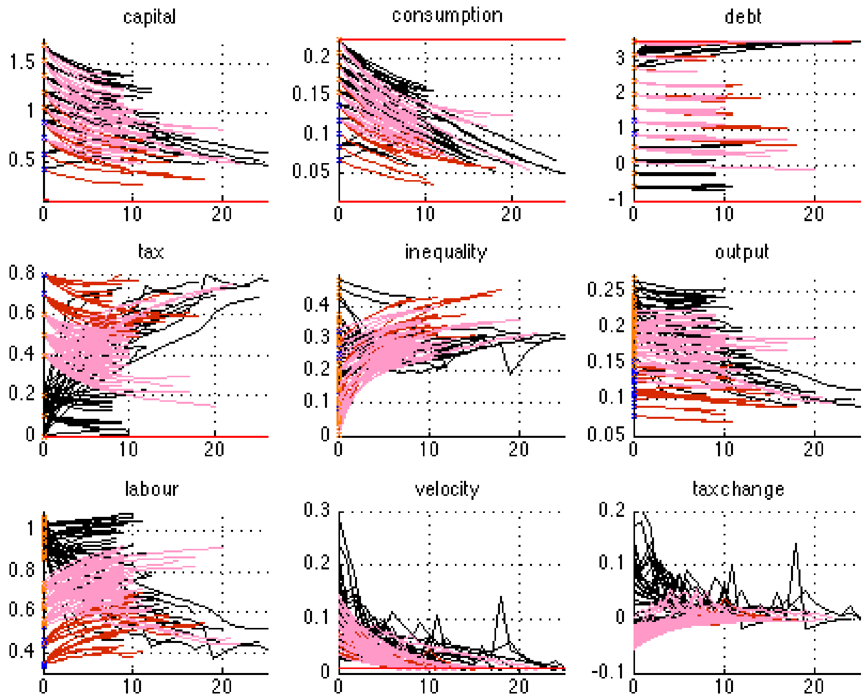

FIGURE 7. Viable evolutions' time profiles for unconstrained χ

The trajectories do not shrink because the kernels' shrinkage is much greater than the shrinkage of the kernels' consumption-capital projections. In Figure 2, the black shapes on the consumption-capital axis include all consumption-capital points which are viable in that kernel. A substantial number of initial economic states become non-viable when a top constraint on χ is imposed, but those points generally have a corresponding viable point with the same capital and consumption. Thus a non-viable point will likely have a viable cousin, perhaps with lower debt or higher tax rates.

The time profiles have been collected into three groups: low initial tax rate (black lines), medium initial tax rate (pink lines) and high initial tax rate (red lines). The high tax category tends to create the

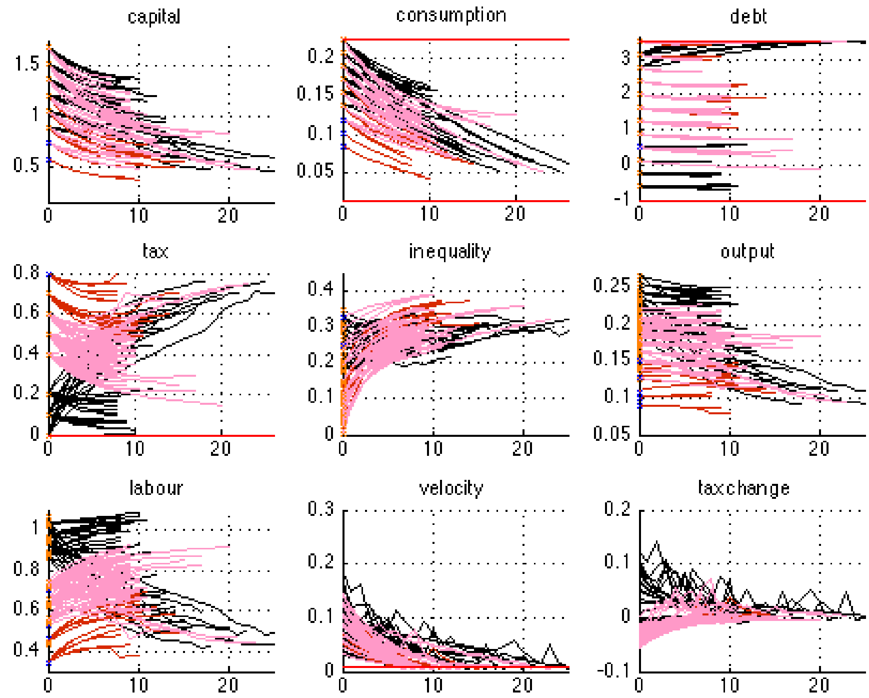

FIGURE 8. Viable evolutions' time profiles for $\chi \leq 0.4$

highest relative factor share and the lowest output. This may be because this category tends to start with low capital. The time profiles with low initial tax rates (black lines) are correlated with low inequality and high output. If initially high taxes (red lines) are to produce viable evolutions with constrained inequality, taxes need be lowered. The medium-high and high taxation paths require labour increases for viability.

We cannot speculate too much about output's impact on inequality or inequality's impact on output: our viable evolutions stabilise at a wide range of steady states which nonetheless have very similar relative factor shares. This does tell us that many distinct steady state levels of output can generate similar levels of inequality.

Our model has diminishing returns to capital and constant total factor productivity, so – unsurprisingly – few of the paths have growing output. Those which do grow tend to have high debt, tax decreases and labour supply increases. As mentioned earlier, a longer-run relationship between output and inequality – such as that captured by the Kuznets curve – would require a model designed for a longer time frame.

6. Reducing an economy's relative factor share

In Section 4 we established the sets of economic states which are sustainable, given different constraints on χ. This section asks how an economy can transition from a high χ state to low χ state.

Figure 9 contains the kernels obtained for $0 \leq \chi$ and $0 \leq \chi \leq 0.4$, marked by the lighter and darker colours respectively. Unsurprisingly, the more constrained kernel is a subset the less constrained kernel. Both kernels have plenty of viable states with low χ.

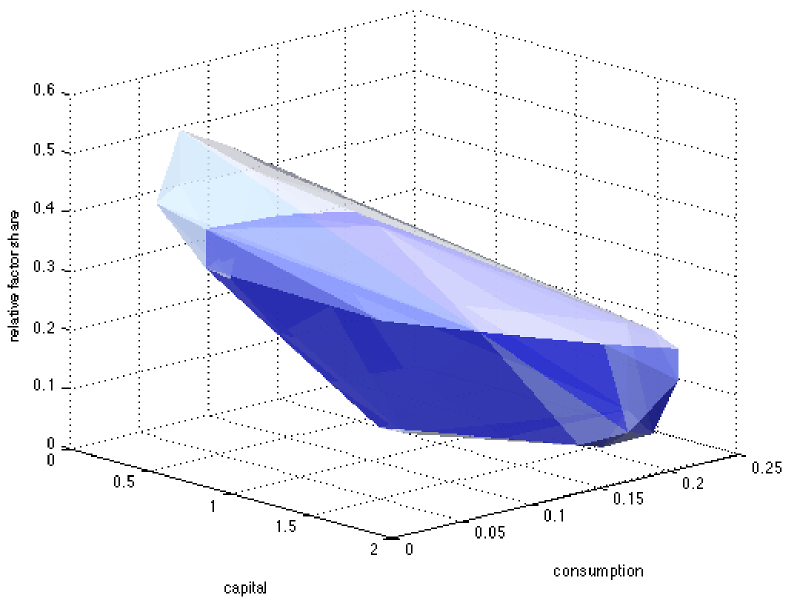

FIGURE 9. The $0 \leq \chi$ and $0 \leq \chi \leq 0.4$ viability kernels

Consider those states within the bigger kernel with $\chi > 0.4$. They are in the top, light-coloured part of the boulder and have low capital and consumption. We want to establish how a social planner could transition an economy from one of these state to one with $\chi \leq 0.4$, sitting in the darker part of the boulder.

Examining the stabilising evolutions which emanate from each viable state reveals the existence of several evolutions which have an intial $\chi > 0.4$ and a stabilised χ below 0.4. Figure 10 shows two of them originating from slightly below $\chi = 0.5$.

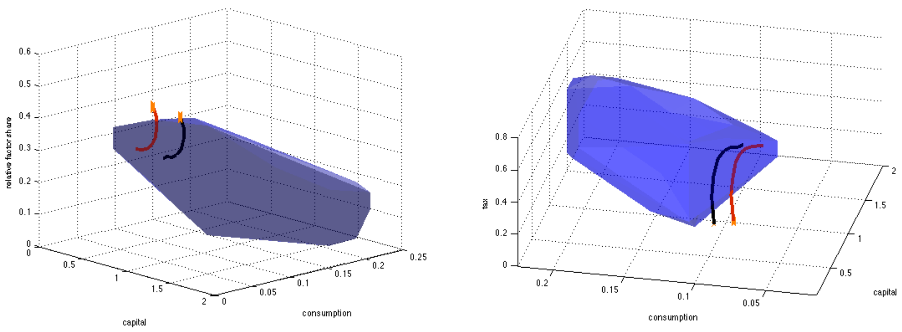

FIGURE 10. Viable evolutions from a high χ economy to $\chi \leq 0.4$. The inequality transition can be observed in the left panel, the taxation transition in the right.

Two different slices of the viability kernel are shown in Figure 10. The slice in the left panel has χ on the vertical axis and so is the same as the darker kernel in Figure 9. A different slice of this viability kernel is shown in the right panel, with tax rates on the vertical axis. The lines show two evolutions from $\chi > 0.4$ to $\chi < 0.4$. We can see that the relative factor share diminishes as the taxation rate rises. In each evolution the capital and consumption vary little.

The evolutions' time profiles are shown in Figure 11. We can see how capital, consumption, debt, tax and χ converge to a near-steady state. This process is a result of controlling the economy by the tax changes

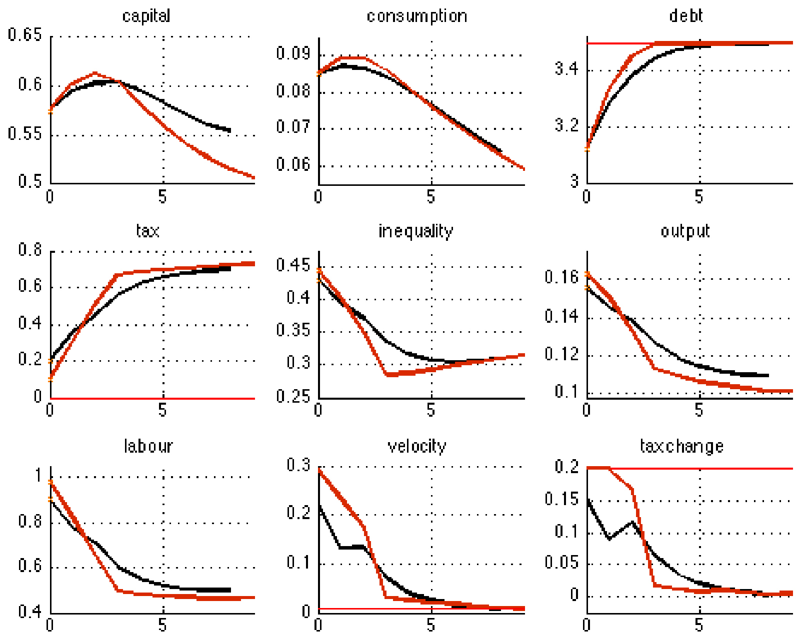

FIGURE 11. Time profiles of viable evolutions from a high χ economy to a constrained $\chi \in [0, 0.4]$

shown in the last panel. Near-steadiness is represented by the diminishing velocity in the low middle panel. Overall, the profiles suggest that tax is not the only state variable which changes substantially. Debt increases fast as tax rates grow while capital and consumption decrese. This is mainly because small tax increments would be too small to keep debt low, when capital decreases.

By and large, by applying tax increases the unequal economy with low tax and low debt has transited to a more equal economy with high taxes and high debt. A more desirable transition could be to a state with higher capital and consumption. However even if such a transition is possible, finding a strategy which generates it requires solving a *crisis-control problem*. This is described in Appendix B.

7. Conclusion

We have analysed which tax-adjustment strategies are compatible with both an efficient economy and low inequality, collectively represented by the constraint set K. We have shown that, for the economies we study, many strategies are compatible: the viability kernel with a top constraint on χ still includes many points – though as the constraint is lowered, the kernel shrinks quickly.

In fact, we showed in Section 5 that most economies will tend to have similar levels of inequality in the long run, despite being substantially different in other ways.

This approach has important implications for policy. For example when capital and consumption are relatively large, low taxes are compatible with low relative factor share and high output. In fact, lowering taxes when the economy has low capital and low consumption seems to bring about a stable economy with low inequality albeit with low output.

Moving aside from policy advice, if we think of our viability theory in positive terms – as a realistic description of how politicians act – we can produce interesting and accurate explanations of economic phenomena. As shown in Section 4, the high taxes and low inequality of Japan, and the low taxes and high inequality of Columbia, are unsurprising given that Japan has a lot of debt and Columbia doesn't.

All that said, more important than our specific results is our demonstration that viability theory is a useful approach to understanding the inescapable trade-offs that our concerns about inequality introduce. In formalising what we consider acceptable we can test whether these considerations are realistic and – if so – how they can be achieved. That will prove immensely valuable to both decision makers and the economists who wish to study them.

References

Alvaredo, F., Atkinson, A.B., Piketty, T., Saez, E., 2014. The world top incomes database. URL: http://topincomes.g-mond.parisschoolofeconomics.eu/. accessed 08/12/2014.

Aubin, J.P., 1991. Viability Theory. Systems & Control: Foundations & Applications, Birkhäuser, Boston. doi:10.1007/978-0-8176-4910-4.

Aubin, J.P., Bayen, A.M., Saint-Pierre, P., 2011. Viability Theory: New Directions. 2 ed., Springer, Berlin. doi:10.1007/978-3-642-16684-6.

Béné, C., Doyen, L., Gabay, D., 2001. A viability analysis for a bio-economic model. Ecological Economics 36, 385–396.

Bonneuil, N., Boucekkine, R., 2008. Sustainability, optimality and viability in the ramsey model. Manuscript, 39 pages.

Bonneuil, N., Saint-Pierre, P., 2008. Beyond optimality: Managing children, assets, and consumption over the life cycle. Journal of Mathematical Economics 44, 227–241.

Clément-Pitiot, H., Doyen, L., 1999. Exchange rate dynamics, target zone and viability.

Clément-Pitiot, H., Saint-Pierre, P., 2006. Goodwin's models through viability analysis: some lights for contemporary political economics regulations, in: 12th International Conference on Computing in Economics and Finance, Cyprus. Conference maker.

De Lara, M., Doyen, L., Guilbaud, T., Rochet, M.J., 2006. Is a management framework based on spawning stock biomass indicators sustainable? A viability approach. ICES Journal of Marine Science .

Doyen, L., Saint-Pierre, P., 1997. Scale of viability and minimal time of crisis. Set-Valued Analysis 5, 227–246.

Gaitsgory, V., Quincampoix, M., 2009. Linear programming approach to deterministic infinite horizon optimal control problems with discounting. SIAM J. Control Optim. 48, 2480–2512. doi:10.1137/070696209.

Krawczyk, J.B., Judd, K.L., 2015. A note on determining viable economic states in a dynamic model of taxation. Macroeconomic Dynamics forthcoming.

Krawczyk, J.B., Kim, K., 2004. A viability theory analysis of a macroeconomic dynamic game, in: Eleventh International Symposium on Dynamic Games and Applications, Tucson, AZ, US.

Krawczyk, J.B., Kim, K., 2009. "Satisficing" solutions to a monetary policy problem: A viability theory approach. Macroeconomic Dynamics 13, 46–80. doi:10.1017/s1365100508070466.

Krawczyk, J.B., Kim, K., 2014. Viable stabilising non-Taylor monetary policies for an open economy. Computational Economics 43, 233–268. doi:10.1007/s10614-013-9360-4.

Krawczyk, J.B., Pharo, A., 2011. Viability Kernel Approximation, Analysis and Simulation Application – VIKAASA Manual. SEF Working Paper 13/2011. Victoria University of Wellington. URL: http://hdl.handle.net/10063/1878.

Krawczyk, J.B., Pharo, A., 2013. Viability theory: an applied mathematics tool for achieving dynamic systems' sustainability. Mathematica Applicanda 41, 97–126. URL: http://wydawnictwa.ptm.org.pl/index.php/matematyka-stosowana/article/view/409.

Krawczyk, J.B., Pharo, A., 2014. Viability Kernel Approximation, Analysis and Simulation Application – VIKAASA Code. URL: http://socsol.github.io/vikaasa/.

Krawczyk, J.B., Pharo, A., Serea, O.S., Sinclair, S., 2013. Computation of viability kernels: a case study of by-catch fisheries. Computational Management Science 10, 365–396. URL: http://dx.doi.org/10.1007/

s10287-013-0189-z, doi:10.1007/s10287-013-0189-z.

Krawczyk, J.B., Serea, O.S., 2013. When can it be not optimal to adopt a new technology? A viability theory solution to a two-stage optimal control problem of new technology adoption. Optimal Control Applications and Methods 34, 127 – 144. URL: http://www.vuw.ac.nz/staff/jacek_krawczyk/somepapers/oca1030.pdf, doi:10.1002/oca.1030.

Krawczyk, J.B., Sethi, R., 2007. Satisficing Solutions for New Zealand Monetary Policy. Technical Report. Reserve Bank of New Zealand, No DP2007/03. URL: http://www.rbnz.govt.nz/research_and_publications/articles/details.aspx?id=3968.

Krawczyk, J.B., Sissons, C., Vincent, D., 2012. Optimal versus satisfactory decision making: a case study of sales with a target. Computational Management Science 9, 233–254. URL: http://www.vuw.ac.nz/staff/jacek_krawczyk/somepapers/cms2012.pdf. published online: February 2012, DOI: 10.1007/s10287-012-0141-7.

Krawczyk, J.B., Townsend, W., 2015a. Basic understanding of social inequality dynamics. Working Paper Series xxxx. Victoria University of Wellington, School of Economics and Finance.

Krawczyk, J.B., Townsend, W., 2015b. New Zealand inequality and the struggle between capital and labour. Working Paper Series xxxx. Victoria University of Wellington, School of Economics and Finance.

Kuznets, S., 1955. Economic growth and income inequality. The American economic review , 1–28.

Martinet, V., Doyen, L., 2007. Sustainability of an economy with an exhaustible resource: A viable control approach. Resource and Energy Economics 29, 17–39.

Martinet, V., Thébaud, O., Doyen, L., 2007. Defining viable recovery paths toward sustainable fisheries. Ecological Economics 64, 411–422. doi:10.1016/j.ecolecon.2007.02.036.

McManus, R., 2013. "We're all in this together"? A DSGE interpretation. Working Paper May. The University of York. URL: http://www.york.ac.uk/media/economics/documents/discussionpapers/2013/1308.pdf.

National Tax Agency, 2010. Income tax guide for foreigners. URL: http://www.nta.go.jp/tetsuzuki/shinkoku/shotoku/tebiki2010/pdf/43.pdf.

Okun, A.M., 1975. Equality and efficiency: The big tradeoff. Washington, DC: The Brookings Institution .

Pujal, D., Saint-Pierre, P., 2006. Capture basin algorithm for evaluating and managing complex financial instruments, in: 12th International Conference on Computing in Economics and Finance, Cyprus. Conference maker.

Quincampoix, M., Veliov, V.M., 1998. Viability with a target: Theory and applications. Applications of Mathematical Engineering , 47–58.

Simon, H.A., 1955. A behavioral model of rational choice. Quarterly Journal of Economics 69, 99–118. doi:10.2307/1884852.

The World Bank, 2015. Quarterly public sector debt. URL: http://data.worldbank.org/data-catalog/quarterly-public-sector-debt.

Veliov, V.M., 1993. Sufficient conditions for viability under imperfect measurement. Set-Valued Analysis, 1, 305–317. doi:10.1007/bf01027640.

Appendix A. A method for finding viability kernels

VIKAASA[6] is a suite of MATLAB® programmes that approximate viability kernels. VIKAASA follows the approach suggested in Gaitsgory and Quincampoix (2009).

VIKAASA can be used either as a set of MATLAB® functions, or via a GUI.[7] The GUI can specify the viability problem, run the kernel approximation algorithms and display the results. A detailed (though somewhat outdated) manual for VIKAASA can be found in Krawczyk and Pharo (2011). The latest version of VIKAASA is available for download at Krawczyk and Pharo (2014). In Figure 12, we show the main window of VIKAASA.

In this paper, our algorithm solves a truncated optimal stabilisation problem for each element of $K^h \subset K$, a discretisation of K. For each $x^h \in K^h$, VIKAASA assesses whether a dynamic evolution originating at x^h can be controlled to a (nearly) steady state without leaving the constraint set in finite time. Those points that can be brought close enough to such a state are included in the kernel while those that are not are excluded. This algorithm (called the *inclusion* algorithm, see Krawczyk et al. (2013)) will miss viable points that cannot reach a steady state, such as those which form orbits.

[6]See Krawczyk and Pharo (2011) and Krawczyk and Pharo (2014); also Krawczyk, Pharo, Serea and Sinclair (2013).

[7]VIKAASA is also compatible with GNU Octave, though its GUI is not.

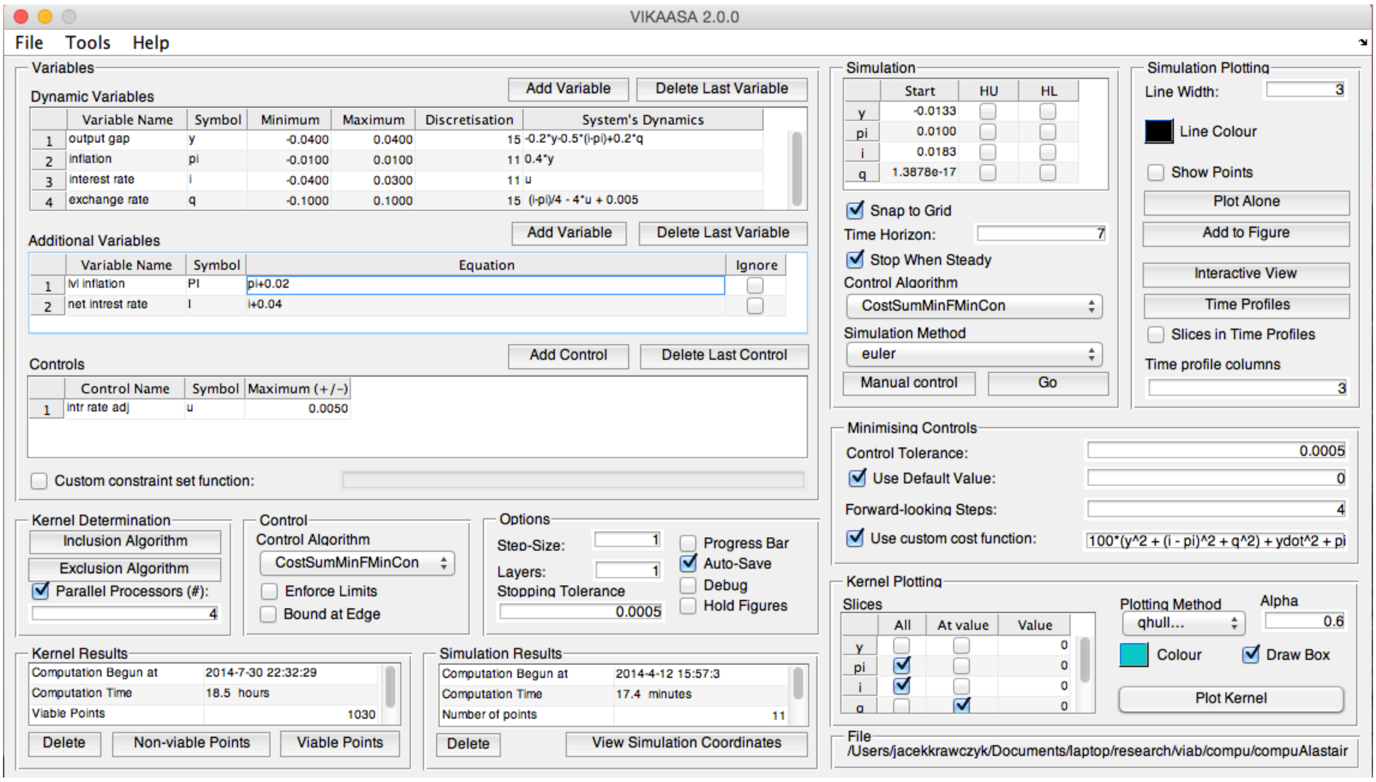

FIGURE 12. VIKAASA main window.

Appendix B. Crisis control

VIKAASA has not shown any evolution of our economy from above $\chi > 0.4$ to $\chi < 0.25$. That is one example of an evolution which, if it exists, would have to be found by solving a *crisis-control problem.*

A crisis-control problem occurs (see Doyen and Saint-Pierre (1997)) when a state of the dynamic system is outside the viability kernel and the planner wants to steer the system into the kernel. The state may be outside the kernel because a shock pushed the system outside the kernel, or because the social planner read this paper too late.

One then needs to seek a crisis-control strategy $u^C(x,t), t \in [0, \Theta]$, where $x = [k, c, B, \tau]$ is the economy state and Θ is finite (possibly minimal) time, as a minimiser of the crisis metric

(11)
$$C(u, x, t; x_{\mathcal{V}}) = \int_0^{\Theta} \frac{1}{2} \left((x(t) - x_{\mathcal{V}})^2 + u(t)^2 \right) dt + \frac{1}{2} \left(x(\Theta) - x_{\mathcal{V}} \right)^2 .$$

So,

$$u^C(x,t) = \arg\min_u C(u, x, t; x_{\mathcal{V}}) \quad (12)$$

where $x_{\mathcal{V}} \in \mathcal{V}_F(K)$ is a target state in $\mathcal{V}_F(K)$.

The constraints on $u(t) \in U$ and $x(t) \in K$ are removed (or at least relaxed) in problem (12). This is because $x \in \mathcal{V}_F(K)$ could not be achieved with $u(t) \in U$.

The control $u^C(x,t)$ that minimises $C(\cdots)$ may not lead $x(t)$ to *exactly* $x_{\mathcal{V}}$. Therefore we will have to check whether if $x(T) \in \mathcal{V}_F(K)$ and, if not, whether the distance between $x_{\mathcal{V}}$ and $x(T)$ is satisfactory.

Furthermore, $u^C(x,t)$ may not exist at *all* because the system's dynamics $F(\cdot)$ is not suffiently controllable.[8] In any case, solving problem (12) is a non-trivial problem of optimal control theory.

[8]This seems to be the case of our system which is highly nonlinear and subject to only a single control.